

Global classical solution to 3D isentropic compressible Navier-Stokes equations with large initial data and vacuum

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Abstract

In this paper, we investigate the existence of a global classical solution to 3D Cauchy problem of the isentropic compressible Navier-Stokes equations with large initial data and vacuum. Precisely, when the far-field density is vacuum ($\tilde{\rho} = 0$), we get the global classical solution under the assumption that $(\gamma - 1)^{\frac{1}{3}} E_0 \mu^{-1}$ is suitably small. In the case that the far-field density is away from vacuum ($\tilde{\rho} > 0$), the global classical solution is also obtained when $\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}}\right) E_0^{\frac{1}{4}} \mu^{-\frac{1}{3}}$ is suitably small. The above results show that the initial energy E_0 could be large if $\gamma - 1$ and $\tilde{\rho}$ are small or the viscosity coefficient μ is taken to be large. These results improve the one obtained by Huang-Li-Xin in [16], where the existence of the classical solution is proved with small initial energy. It should be noted that in the theorems obtained in this paper, no smallness restriction is put upon the initial data. It can be viewed the first result on the existence of the global classical solution to three-dimensional Navier-Stokes equations with large initial energy and vacuum when γ is near 1.

Key Words: Compressible Navier-Stokes equations, global classical solution, vacuum.

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1 Introduction

In this paper, we consider the following isentropic compressible Navier-Stokes system in three-dimensional space

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u, \end{cases} \quad x \in \mathbb{R}^3, \quad t > 0, \quad (1.1)$$

with the initial condition

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3, \quad (1.2)$$

and the far-field behavior

$$\rho(x, t) \rightarrow \tilde{\rho} \geq 0, \quad u(x, t) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \quad \text{for } t \geq 0. \quad (1.3)$$

Here $\rho = \rho(x, t)$ and $u = u(x, t) = (u_1, u_2, u_3)(x, t)$ represent the density and velocity of the fluid respectively; the pressure P is given by

$$P(\rho) = A\rho^\gamma,$$

where $\gamma > 1$ is the adiabatic exponent, $A > 0$ is a constant. The constant viscosity coefficients μ and λ satisfy the following physical restrictions

$$\mu > 0, \quad \lambda + \frac{2\mu}{3} \geq 0. \quad (1.4)$$

A great number of works have been devoted to the well-posedness of solutions to Navier-Stokes equations. The one-dimensional problem has been studied in many papers, for example [6, 20, 25, 27, 33, 35, 41, 42, 44] and so on. For the multi-dimensional case, in the absence of vacuum, the local existence and uniqueness of classical solutions are known in [28, 31]. Matsumura and Nishida in [26] first proved the global existence and uniqueness of classical solutions for the initial data close to a non-vacuum equilibrium in some Sobolev space H^s . Later, the global existence of weak solutions was proved by Hoff [9, 10] for the discontinuous initial data with small energy. In the presence of vacuum, due to the degeneration of momentum equation, the issue becomes much more challenging. The global existence of weak solutions with large initial data in \mathbb{R}^N was first obtained by Lions in [23] for $\gamma \geq \frac{3N}{N+2}$ ($N = 2, 3$), where the initial energy is finite, so that the density vanishes at far fields or even has compact support. Later, E. Feireisl, A. Novotny and H. Petzeltov in [8] extended Lions's result to the case $\gamma > \frac{3}{2}$ for $N = 3$, which include the monoatomic gas $\gamma = \frac{5}{3}$. Jiang and Zhang in [17, 18] proved the global existence of weak solutions with vacuum for any $\gamma > 1$ for spherical symmetry or axisymmetric initial data. However, the regularity and uniqueness of weak solutions are basically open in general. Recently, when the far-field density is away from vacuum ($\tilde{\rho} > 0$) and the viscosity coefficients μ and λ satisfy the assumption that $\mu > \max\{4\lambda, -\lambda\}$, Hoff and associates in [11, 12, 13] obtained a new type of global weak solutions with small energy, which have extra regularity compared with those large weak ones constructed by Lions ([23]) and Feireisl et al. ([8]).

In spite of the huge amount of work, it is still a major open problem whether or not global (strong) classical solutions exist in three space dimensions for general initial data with vacuum. The local existence and uniqueness of (strong) classical solutions with vacuum are known in [1, 2, 3]. It seems that one should not expect better regularities of the global solutions in general due to Xin's results ([39]) and Rozanova's results ([30]). It was proved

that there is no global smooth solution in $C^1([0, \infty); H^m(\mathbb{R}^d))$ ($m > [\frac{d}{2}] + 2$) to the Cauchy problem of the full compressible Navier-Stokes system, if the initial density is nontrivial compactly supported ([39]), or the solutions are highly decreasing at infinity ([30]). Xin and Yan in [40] improved the blow-up results in [39] by removing the assumptions that the initial density has compact support and the smooth solution has finite energy.

More recently, Huang-Li-Xin in [16] established the surprising global existence and uniqueness of classical solutions with constant state as far field which could be either vacuum or nonvacuum to 3D isentropic compressible Navier-Stokes equations with small total energy but possibly large oscillations. Then a natural question of importance and interest is how to get a global classical solution for large initial energy. In this paper, we devote ourselves to this problem and show that, such a large classical solution to (1.1)-(1.3) indeed exists globally, when γ is near 1 or μ is taken to be large.

Before stating our main results, we would like to give some notations which will be used throughout this paper.

Notations:

(i) $\int_{\mathbb{R}^3} f = \int_{\mathbb{R}^3} f dx, \quad \int_0^T g = \int_0^T g dt.$

(ii) For $1 \leq l \leq \infty$, denote the L^l spaces and the standard Sobolev spaces as follows:

$$L^l = L^l(\mathbb{R}^3), \quad D^{k,l} = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3) : \|\nabla^k u\|_{L^l} < \infty \right\}, \quad D^k = D^{k,2},$$

$$W^{k,l} = L^l \cap D^{k,l}, \quad H^k = W^{k,2}, \quad D_0^1 = \left\{ u \in L^6 : \|\nabla u\|_{L^2} < \infty \right\},$$

$$\dot{H}^\beta = \left\{ u : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad \|u\|_{\dot{H}^\beta}^2 = \int |\xi|^{2\beta} |\hat{u}(\xi)|^2 d\xi < \infty \right\}.$$

(iii) $G = (2\mu + \lambda)\text{div} u - P$ is the effective viscous flux.

(iv) $\omega = \nabla \times u$ is the vorticity.

(v) $\dot{h} = h_t + u \cdot \nabla h$ denotes the material derivative.

(vi) $\sigma = \sigma(t) = \min\{1, t\}$.

(vii) $E_0 = \int_{\mathbb{R}^3} \left(\frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right)$ is the initial energy, where G denotes the potential energy density given by

$$G(\rho) \triangleq \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds.$$

It is clear that

$$\begin{cases} G(\rho) = \frac{1}{\gamma - 1} P & \text{if } \tilde{\rho} = 0, \\ \frac{1}{c(\bar{\rho}, \tilde{\rho})} (\rho - \tilde{\rho})^2 \leq G(\rho) \leq c(\bar{\rho}, \tilde{\rho}) (\rho - \tilde{\rho})^2 & \text{if } \tilde{\rho} > 0, \quad 0 \leq \rho \leq \bar{\rho}, \end{cases}$$

for some positive constant $c(\bar{\rho}, \tilde{\rho})$.

Now we state our main results. One of our main results is the following global existence to (1.1)-(1.3) with vacuum at infinite ($\tilde{\rho} = 0$).

Theorem 1.1 *For any given $M > 0$ (not necessarily small) and $\bar{\rho} \geq 1$, assume that the initial data (ρ_0, u_0) satisfy*

$$\frac{1}{2}\rho_0|u_0|^2 + \frac{A}{\gamma-1}\rho_0^\gamma \in L^1, \quad u_0 \in D^1 \cap D^3, \quad (\rho_0, P(\rho_0)) \in H^3, \quad (1.5)$$

$$0 \leq \rho_0 \leq \bar{\rho}, \quad \|\nabla u_0\|_{L^2}^2 \leq M \quad (1.6)$$

and the compatibility condition

$$-\mu\Delta u_0 - (\mu + \lambda)\nabla \operatorname{div} u_0 + \nabla P(\rho_0) = \rho_0 g, \quad (1.7)$$

where $g \in D^1$ and $\rho^{\frac{1}{2}}g \in L^2$. Then there exists a unique global classical solution (ρ, u) in $\mathbb{R}^3 \times [0, \infty)$ satisfying, for any $0 < \tau < T < \infty$,

$$0 \leq \rho \leq 2\bar{\rho}, \quad x \in \mathbb{R}^3, \quad t \geq 0, \quad (1.8)$$

$$\begin{cases} (\rho, P) \in C([0, T]; H^3), & \sqrt{\rho}u_t \in L^\infty(0, T; L^2), \\ u \in C([0, T]; D^1 \cap D^3) \cap L^2(0, T; D^4) \cap L^\infty(\tau, T; D^4), \\ u_t \in L^\infty(0, T; D^1) \cap L^2(0, T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1), \end{cases} \quad (1.9)$$

provided that

$$\frac{(\gamma-1)^{\frac{1}{3}}E_0}{\mu} \leq \varepsilon \triangleq \min \left\{ \varepsilon_3^2, (2C(\bar{\rho}, M))^{-\frac{32}{3}}\mu^8, (4C(\bar{\rho}))^{-4} \right\}, \quad (1.10)$$

where

$$\begin{aligned} \varepsilon_3 &= \min \left\{ (CE_7)^{-3} \Big|_{(1 < \gamma \leq \frac{3}{2})}, (CE_{11})^{-2} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_2 \right\}, \\ \varepsilon_2 &= \min \left\{ C(\bar{\rho})^{-2}(\gamma-1)^{-\frac{2}{3}}E_2^{-3}\mu^5 \Big|_{(1 < \gamma \leq \frac{3}{2})}, C(\bar{\rho})^{-1}\mu^{\frac{9}{4}}E_2^{-\frac{3}{4}} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_1 \right\}, \\ \varepsilon_1 &= \min \left\{ \left(4C(\bar{\rho})\right)^{-6}, 1 \right\}. \end{aligned}$$

Here, C depending on $\bar{\rho}, M$ and some other known constants but independent of $\mu, \lambda, \gamma-1$ and t (see (3.71), (3.74)). E_2, E_7 and E_{11} are defined by (3.29), (3.53) and (3.65) respectively.

Now we briefly outline the main ideas of the proof of Theorem 1.1, some of which are inspired by [9] and [16]. The key point of this paper is to get the time-independent upper bound on the density ρ (see (3.68)), and once that is obtained, the proof of Theorem 1.1 follows in the same way as in [16]. It's worth noting that the methods in all previous works [9, 16, 43] depend crucially on the small initial energy. Thus, some new ideas are needed to recover all the *a priori* estimates under only the assumption (1.10) while the initial energy E_0 could be large (see (3.40)-(3.54)). In fact, the small initial energy could naturally yield the smallness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$, which plays a crucial role in the analysis to prove the time-independent upper bound of ρ . But the smallness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$ is not valid here without the small initial energy. The crucial ideas to overcome this difficulty are as follows:

• One key observation is that $A_1(T)$ could be bounded by $\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2$ and some other terms, i.e.,

$$A_1(T) \leq \dots + \frac{(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + \frac{C(\gamma - 1)E_0}{\mu^2}. \quad (\text{see (3.10)})$$

Thus, we can close the *a priori* assumption on $A_1(T)$ by the smallness of $\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2$ instead of the smallness of $\int_0^T \|\nabla u\|_{L^2}^2$. And we can prove that $\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2$ is small, when $(\gamma - 1)^{\frac{1}{3}} E_0 \mu^{-1}$ is suitably small (see (3.7)).

• Another new ingredient in the proof is that we can use the smallness of $(\gamma - 1)^{\frac{1}{3}} E_0 \mu^{-1}$ and the boundedness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$ to estimate the higher-order terms of ∇u as follows:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 &\leq CN_4^4 (2\mu + \lambda)^{-3} \int_0^T \sigma^2 \|\rho \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2} + \dots \\ &\leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{1}{2}} A_2^{\frac{1}{2}}(T) \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho\|_{L^3}^2 \|\nabla \dot{u}\|_{L^2}^2 \\ &\quad + \dots \\ &\leq (2\mu + \lambda)^{-3} \mu^{\frac{1}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6 \quad (\text{see (3.40) - (3.47)}) \end{aligned}$$

and

$$\begin{aligned} \int_0^T \sigma \|\nabla u\|_{L^3}^3 &\leq \left(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{2}} \\ &\leq C \frac{(\gamma - 1)^{\frac{3}{8}} E_0^{\frac{23}{24}} E_6^{\frac{1}{2}}}{\mu^{\frac{3}{8}} (2\mu + \lambda)^{\frac{3}{2}}}. \quad (\text{see (3.48)}) \end{aligned}$$

We refer the details to Lemma 3.8.

On the other hand, from the proof of Proposition 3.1, we know that it is important to find a suitable match for μ , $(\gamma - 1)$ and E_0 . That means much more complicated estimates than those in [9, 16, 43] are needed. To do this, we derive some more sophisticated inequalities about μ (see Lemma 2.2). For more details, please see the proof of Proposition 3.1.

Concerning the global classical solutions for (1.1)-(1.3) in the case that the far-field density is away from vacuum ($\tilde{\rho} > 0$), we have

Theorem 1.2 *For any given $M > 0$ (not necessarily small) and $\bar{\rho} \geq \tilde{\rho} + 1$, assume that the initial data (ρ_0, u_0) satisfy*

$$\frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \in L^1, \quad u_0 \in H^1 \cap D^3, \quad (\rho_0 - \tilde{\rho}, P(\rho_0) - P(\tilde{\rho})) \in H^3, \quad (1.11)$$

$$0 \leq \rho_0 \leq \bar{\rho}, \quad \|u_0\|_{L^2}^2 \leq E_0, \quad \|\nabla u_0\|_{L^2}^2 \leq M \quad (1.12)$$

and the compatibility condition

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \operatorname{div} u_0 + \nabla P(\rho_0) = \rho_0 g, \quad (1.13)$$

where $g \in D^1$ and $\rho^{\frac{1}{2}}g \in L^2$. Then there exists a unique global classical solution (ρ, u) in $\mathbb{R}^3 \times [0, \infty)$ satisfying, for any $0 < \tau < T < \infty$,

$$0 \leq \rho \leq 2\bar{\rho}, \quad x \in \mathbb{R}^3, \quad t \geq 0, \quad (1.14)$$

$$\begin{cases} (\rho - \tilde{\rho}, P - P(\tilde{\rho})) \in C([0, T]; H^3), & \sqrt{\rho}u_t \in L^\infty(0, T; L^2), \\ u \in C([0, T]; D^1 \cap D^3) \cap L^2(0, T; D^4) \cap L^\infty(\tau, T; D^4), \\ u_t \in L^\infty(0, T; D^1) \cap L^2(0, T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1), \end{cases} \quad (1.15)$$

provided that $\frac{(\gamma - 1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \frac{\tilde{\rho}}{2C}$ and

$$\frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}}\right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \varepsilon \triangleq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}, \quad (1.16)$$

where

$$\varepsilon_6 = \min \left\{ \left(C(E_{18} + E_{19} + E_{20}) \right)^{-17}, \left(C(E_{18} + E_{19} + E_{21}) \right)^{-8}, \varepsilon_5 \right\},$$

$$\varepsilon_5 = \min \left\{ \left(C(E_{15}E_{17} + E_{16}) \right)^{-4}, \varepsilon_4 \right\},$$

$$\varepsilon_4 = \min \left\{ \left(4C(\bar{\rho}) \right)^{-6}, 1 \right\}.$$

Here C denotes a generic positive constant depending on $\bar{\rho}, M$ and some other known constants but independent of $\mu, \lambda, \gamma - 1, \tilde{\rho}$, and t . $E_{15} - E_{21}$ are defined by (4.40), (4.62), (4.64), (4.66) and (4.74).

There are two key technical points in the proof of Theorem 1.2:

(1) The smallness of $\|\rho - \tilde{\rho}\|_{L^2}$ plays a key role in establishing the time-independent upper bound for the density ρ . In [9, 16, 43], the smallness of $\|\rho - \tilde{\rho}\|_{L^2}$ is easy to get because of the small initial energy. And in the proof of Theorem 1.1 where $\tilde{\rho} = 0$, $\|\rho\|_{L^2}$ can be small when γ is near 1. But, when $\tilde{\rho} > 0$ and the initial energy is large, it seems impossible to get the smallness of $\|\rho - \tilde{\rho}\|_{L^2}$, even as $\gamma \rightarrow 1$. Based on the elaborate analysis of the potential energy density $G(\rho)$, we succeed in deriving a new estimate to $\rho - \tilde{\rho}$, i.e.,

$$\begin{aligned} G(\rho) &= \frac{1}{\gamma - 1} [\rho^\gamma - \tilde{\rho}^\gamma - \gamma \tilde{\rho}^{\gamma-1} (\rho - \tilde{\rho})] \\ &\geq \begin{cases} (\gamma - 1)^{-\frac{1}{4}} |\rho - \tilde{\rho}|^{\gamma-1}, & |\rho - \tilde{\rho}| > (\gamma - 1)^{\frac{1}{3}}, \\ (\gamma - 1)^{-\frac{1}{4}} |\rho - \tilde{\rho}|^3, & |\rho - \tilde{\rho}| \leq (\gamma - 1)^{\frac{1}{3}}, \end{cases} \end{aligned}$$

which implies the L^3 -norm of $\rho - \tilde{\rho}$ can be small when $\gamma \rightarrow 1$ (see Lemma 4.2). In fact, the L^3 -norm of $\rho - \tilde{\rho}$ is weaker than the L^2 -norm in the whole space, when ρ is upper-bounded. Thus, applying the smallness of L^3 -norm of $\rho - \tilde{\rho}$ to establish the time-independent upper bound for the density ρ will bring much more difficulties than that of L^2 -norm.

- Unlike Theorem 1.1, the method in obtaining the smallness of $\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2$ is no longer applicable here. To overcome this difficulty, we have to give the estimate of $\|u\|_{L^2}$ (see (4.12)-(4.13)). This together with some other estimates yields the smallness of $\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2$ (see Lemma 4.3).

- All the estimates containing $P - P(\tilde{\rho})$ in the proof of Theorem 1.2 can not be estimated as the previous way. Hence, some new ideas are needed to recover all the *a priori* estimates under the condition that $\|\rho - \tilde{\rho}\|_{L^3}$ is small.

(2) Due to losing the smallness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$ and $\|\rho - \tilde{\rho}\|_{L^2}$, the techniques in [9, 16, 43] which are used to close the *a priori* assumptions on $A_1(T)$ and $A_2(T)$ fail in this paper. From (4.70) (see N_{11})

$$A_1(T) \leq \cdots + \frac{CE_0^{\frac{1}{2}}}{\mu^2} A_1^{\frac{3}{4}}(T) A_2^{\frac{1}{4}}(T) + \cdots ,$$

we find that, in order to close the *a priori* assumption on $A_1(T)$, the bound of $A_2(T)$ should be given a higher order than that of $A_1(T)$ (Roughly speaking, we can choose $A_2(T) \sim A_1(T)^{\frac{8}{3}}$ in the present paper to close the *a priori* assumptions on $A_1(T)$ and $A_2(T)$). This means we could not close them simultaneously. Fortunately, we observe that $A_2(T)$ could be bounded by the boundedness of $A_1(\sigma(T))$ and some other terms, i.e.,

$$A_2(T) \leq CA_1(\sigma(T)) + \cdots + \frac{CP(\tilde{\rho})^2}{\mu^2} E_0. \quad (\text{see (4.46)})$$

And we can give a better estimate of $A_1(\sigma(T))$ compared with $A_1(T)$'s. This is based on that $\int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^3$ has a stronger control than $\int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^3}^3$ (see (4.56) and (4.68)). Therefore, to close the *a priori* assumptions on $A_1(T)$ and $A_2(T)$, the first step is to estimate $A_1(\sigma(T))$ (see (4.58)). Next, we can bound $A_2(T)$ in (4.60). Finally, the estimate of $A_1(T)$ is obtained in (4.70).

Remark 1.3 *This should be the first result concerning the global existence and uniqueness of classical solutions to the Cauchy problem for the three dimensional isentropic compressible Navier-Stokes equations with large initial energy and vacuum. Compared to these results in [9, 16], the thrust of this paper is to remove the condition of smallness on the initial energy.*

Remark 1.4 *The results in this paper generalize the ones in [16]. More accurately, in the case of $\tilde{\rho} = 0$ and $\tilde{\rho} > 0$, the existence of classical solutions to (1.1)-(1.3) are obtained respectively; in particular, the initial energy is allowed to be large when $\gamma - 1$ and $\tilde{\rho}$ are suitable small or μ is taken to be large. On the other hand, Theorems 1.1 and 1.2 are still applicable to the case that initial energy E_0 is small for any given γ , $\tilde{\rho}$ and μ .*

Remark 1.5 *It is well known that Nishida and Smoller in [29] proved the Cauchy problem for 1D isentropic Euler equations has a global solution provided that $(\gamma - 1)\text{T.V.}\{u_0, \rho_0\}$ is sufficiently small. This means that when γ is near 1, one can allow large data, and conversely, as γ increases, one must take correspondingly smaller data. Recently, Tan-Yang-Zhao-Zou in [34] obtained a global smooth solution to the one-dimensional compressible Navier-Stokes-Poisson equations with large initial data under the assumption that γ is near 1. Soon later, a similar result on 1D compressible Navier-Stokes equations was obtained by Liu-Yang-Zhao-Zou in [24]. These works inspire us to look for the large solution to (1.1)-(1.3) when γ is near 1.*

Remark 1.6 *From physical viewpoint, it is very nature to get a global large solution to (1.1)-(1.3), when the viscosity coefficient is sufficiently large. Note that the coefficients of viscosity are only required to satisfy the physical restriction (1.4) in the present paper.*

Remark 1.7 *Though the initial data can be large if the adiabatic exponent γ goes to 1 or the viscosity coefficient μ is taken to be large, it is still unknown whether the global classical solution exists when the initial data is large for any fixed γ and μ . It should be noted that the similar question of whether there exists a global smooth solution of the three-dimensional incompressible Navier-Stokes equations with smooth initial data is one of the most outstanding mathematical open problems ([7]). Motivated by this, some blow-up criterions of strong (classical) solutions to (1.1) have been studied, please refer for instance to [14, 15, 32, 38] and references therein. In fact, for initial-boundary-value problems or periodic problems of compressible Navier-Stokes equations with vacuum in one dimension, or in two dimensions for isentropic flow, or in higher dimensions with symmetric initial data, the existence of global large regular solutions has been obtained, please refer to [4, 5, 19, 36, 37] and references therein.*

Remark 1.8 *In addition to the conditions of Theorem 1.1 and Theorem 1.2, if we assume further that $u_0 \in \dot{H}^\beta$ ($\beta \in (\frac{1}{2}, 1]$) and replace $\|\nabla u_0\|_{L^2}^2 \leq M$ with $\|u_0\|_{\dot{H}^\beta} \leq \tilde{M}$, the conclusions in Theorem 1.1 and Theorem 1.2 will still hold, and the ε will also depend on \tilde{M} instead of M correspondingly. This can be achieved by a similar way as in [16].*

The rest of the paper is organized as follows. In Section 2, we collect some known inequalities and facts which will be frequently used later. In Section 3, we obtain the prove of Theorem 1.1. Then the proof of Theorem 1.2 is completed in Section 4.

2 Preliminaries

If the solutions are regular enough (such as strong solutions and classical solutions), (1.1) is equivalent to the following system

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u. \end{cases} \quad (2.1)$$

System (2.1) is supplemented with initial condition

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3, \quad (2.2)$$

and the far-field behavior

$$\rho(x, t) \rightarrow \tilde{\rho} \geq 0, \quad u(x, t) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \quad \text{for } t \geq 0. \quad (2.3)$$

Since the exact value of A in the pressure P doesn't play a role in this paper, we henceforth assume $A = 1$. Next, we will list several facts which will be used in the proof of the main results. The first one is the well-known Gagliardo-Nirenberg inequality (see [21]).

Lemma 2.1 *For any $p \in [2, 6]$, $q \in (1, \infty)$ and $r \in (3, \infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for $f \in H^1(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$, we have*

$$\|f\|_{L^p(\mathbb{R}^3)}^p \leq C \|f\|_{L^2(\mathbb{R}^3)}^{\frac{6-p}{2}} \|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{3p-6}{2}}, \quad (2.4)$$

$$\|g\|_{C(\mathbb{R}^3)} \leq C \|g\|_{L^q(\mathbb{R}^3)}^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|_{L^r(\mathbb{R}^3)}^{\frac{3r}{3r+q(r-3)}}. \quad (2.5)$$

We now state some elementary estimates that follow from (2.4) and the standard L^p -estimate for the following elliptic system derived from the momentum equation in (1.1):

$$\Delta G = \operatorname{div}(\rho \dot{u}), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u}). \quad (2.6)$$

Lemma 2.2 *Let (ρ, u) be a smooth solution of (2.1), (2.3). Then there exists a generic positive constant C such that for any $p \in [2, 6]$*

$$\|\nabla G\|_{L^p} \leq C \|\rho \dot{u}\|_{L^p}, \quad \|\nabla \omega\|_{L^p} \leq \frac{C}{\mu} \|\rho \dot{u}\|_{L^p}, \quad (2.7)$$

$$\|G\|_{L^p} \leq C \left((2\mu + \lambda) \|\nabla u\|_{L^2} + \|P - P(\tilde{\rho})\|_{L^2} \right)^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}, \quad (2.8)$$

$$\|G\|_{L^p} \leq C \left((2\mu + \lambda) \|\nabla u\|_{L^3} + \|P - P(\tilde{\rho})\|_{L^3} \right)^{\frac{6-p}{p}} \|\rho \dot{u}\|_{L^2}^{\frac{2p-6}{p}}, \quad (2.9)$$

$$\|w\|_{L^p} \leq C \left(\frac{1}{\mu} \right)^{\frac{3p-6}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}, \quad (2.10)$$

$$\begin{aligned} \|\nabla u\|_{L^p} &\leq C \left((2\mu + \lambda) \|\nabla u\|_{L^3} + \|P - P(\tilde{\rho})\|_{L^3} \right)^{\frac{6-p}{p}} \|\rho \dot{u}\|_{L^2}^{\frac{2p-6}{p}} \\ &\quad + C \|P - P(\tilde{\rho})\|_{L^p} + C \|w\|_{L^p} \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \|\nabla u\|_{L^p} &\leq C N_p (2\mu + \lambda)^{\frac{6-3p}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \\ &\quad + \frac{C}{2\mu + \lambda} \left(\|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \|P - P(\tilde{\rho})\|_{L^2}^{\frac{6-p}{2p}} + \|P - P(\tilde{\rho})\|_{L^p} \right), \end{aligned} \quad (2.12)$$

where $N_p = 1 + \left(2 + \frac{\lambda}{\mu} \right)^{\frac{3p-6}{2p}}$.

Proof. The standard L^p -estimate for elliptic system (2.6) yields (2.7). Using (2.4), we obtain

$$\begin{aligned} \|G\|_{L^p} &\leq C \|G\|_{L^2}^{\frac{6-p}{2p}} \|G\|_{L^6}^{\frac{3p-6}{2p}} \leq C \|G\|_{L^2}^{\frac{6-p}{2p}} \|\nabla G\|_{L^2}^{\frac{3p-6}{2p}} \\ &\leq C \left((2\mu + \lambda) \|\nabla u\|_{L^2} + \|P - P(\tilde{\rho})\|_{L^2} \right)^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} \|G\|_{L^p} &\leq C \|G\|_{L^3}^{\frac{6-p}{p}} \|G\|_{L^6}^{\frac{2p-6}{p}} \leq C \|G\|_{L^3}^{\frac{6-p}{p}} \|\nabla G\|_{L^2}^{\frac{2p-6}{p}} \\ &\leq C \left((2\mu + \lambda) \|\nabla u\|_{L^3} + \|P - P(\tilde{\rho})\|_{L^3} \right)^{\frac{6-p}{p}} \|\rho \dot{u}\|_{L^2}^{\frac{2p-6}{p}}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \|w\|_{L^p} &\leq C \|w\|_{L^2}^{\frac{6-p}{2p}} \|w\|_{L^6}^{\frac{3p-6}{2p}} \leq C \|w\|_{L^2}^{\frac{6-p}{2p}} \|\nabla w\|_{L^2}^{\frac{3p-6}{2p}} \\ &\leq C \left(\frac{1}{\mu} \right)^{\frac{3p-6}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}. \end{aligned} \quad (2.15)$$

Note that $-\Delta u = -\nabla \operatorname{div} u + \nabla \times \omega$, which implies that

$$\nabla u = -\nabla(-\Delta)^{-1} \nabla \operatorname{div} u + \nabla(-\Delta)^{-1} \nabla \times \omega.$$

Thus the standard L^p -estimate shows that

$$\|\nabla u\|_{L^p} \leq C (\|\operatorname{div} u\|_{L^p} + \|\omega\|_{L^p}) \quad \text{for } p \in [2, 6],$$

which together with (2.8), (2.10) and the definition of G , give

$$\begin{aligned} \|\nabla u\|_{L^p} &\leq C \|\operatorname{div} u\|_{L^p} + C \|\operatorname{curl} u\|_{L^p} \\ &\leq \frac{C}{2\mu + \lambda} \left(\|G\|_{L^p} + \|P - P(\tilde{\rho})\|_{L^p} \right) + C \|w\|_{L^p} \\ &\leq C \left((2\mu + \lambda)^{\frac{6-3p}{2p}} + \mu^{\frac{6-3p}{2p}} \right) \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \\ &\quad + \frac{C}{2\mu + \lambda} \left(\|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \|P - P(\tilde{\rho})\|_{L^2}^{\frac{6-p}{2p}} + \|P - P(\tilde{\rho})\|_{L^p} \right) \\ &\leq C N_p (2\mu + \lambda)^{\frac{6-3p}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \\ &\quad + \frac{C}{2\mu + \lambda} \left(\|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \|P - P(\tilde{\rho})\|_{L^2}^{\frac{6-p}{2p}} + \|P - P(\tilde{\rho})\|_{L^p} \right). \end{aligned} \quad (2.16)$$

□

Lemma 2.3 ([45]) *Let the function y satisfy*

$$y'(t) = g(y) + b'(t) \quad \text{on } [0, T], \quad y(0) = y^0,$$

with $g \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1),$$

for all $0 \leq t_1 \leq t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then

$$y(t) \leq \max\{y^0, \bar{\xi}\} + N_0 < \infty \quad \text{on } [0, T],$$

where $\bar{\xi}$ is a constant such that

$$g(\xi) \leq -N_1, \quad \text{for } \xi \geq \bar{\xi}.$$

3 The proof of Theorem 1.1

In this section, we will first establish the time-independent upper bound of the density ρ . Assume that (ρ, u) is a smooth solution to (1.1)-(1.3) on $\mathbb{R}^3 \times (0, T)$ for some positive time $T > 0$. Set $\sigma = \sigma(t) = \min\{1, t\}$ and denote

$$\begin{cases} A_1(T) = \sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \sigma \frac{\rho |\dot{u}|^2}{\mu}, \\ A_2(T) = \sup_{0 \leq t \leq T} \sigma^2 \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla \dot{u}|^2, \\ A_3(T) = \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \frac{\rho |u|^3}{\mu^3}. \end{cases} \quad (3.1)$$

The following proposition plays a crucial role in this section.

Proposition 3.1 Assume that the initial data satisfies (1.5), (1.6) and (1.7). If the solution (ρ, u) satisfies

$$A_1(T) + A_2(T) \leq \frac{2(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \quad A_3(\sigma(T)) \leq \frac{2(\gamma - 1)^{\frac{1}{12}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{4}}}, \quad 0 \leq \rho \leq 2\bar{\rho}, \quad (3.2)$$

then for any $(x, t) \in \mathbb{R}^3 \times [0, T]$, the following estimates hold:

$$A_1(T) + A_2(T) \leq \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \quad A_3(\sigma(T)) \leq \frac{(\gamma - 1)^{\frac{1}{12}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{4}}}, \quad 0 \leq \rho \leq \frac{7}{4}\bar{\rho}, \quad (3.3)$$

provided $\frac{(\gamma - 1)^{\frac{1}{3}} E_0}{\mu} \leq \varepsilon$. Here

$$\varepsilon = \min \left\{ \varepsilon_3^2, (2C(\bar{\rho}, M))^{-\frac{32}{3}} \mu^8, (4C(\bar{\rho}))^{-4} \right\},$$

and

$$\begin{aligned} \varepsilon_3 &= \min \left\{ (CE_7)^{-3} \Big|_{(1 < \gamma \leq \frac{3}{2})}, (CE_{11})^{-2} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_2 \right\}, \\ \varepsilon_2 &= \min \left\{ C(\bar{\rho})^{-2} (\gamma - 1)^{-\frac{2}{3}} E_2^{-3} \mu^5 \Big|_{(1 < \gamma \leq \frac{3}{2})}, C(\bar{\rho})^{-1} \mu^{\frac{9}{4}} E_2^{-\frac{3}{4}} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_1 \right\}, \\ \varepsilon_1 &= \min \left\{ (4C(\bar{\rho}))^{-6}, 1 \right\}. \end{aligned}$$

Here, C depending on $\bar{\rho}, M$ and some other known constants but independent of $\mu, \lambda, \gamma - 1$ and t (see (3.71), (3.74)). E_2, E_7 and E_{11} are defined by (3.29), (3.53) and (3.65) respectively.

Proof. Proposition 3.1 can be derived from Lemmas 3.2-3.9 below.

Lemma 3.2 Under the conditions of Proposition 3.1, it holds that

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} P \leq (\gamma - 1) E_0, \quad (3.4)$$

$$\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{E_0}{\mu}. \quad (3.5)$$

Proof. Multiplying (2.1)₁ by $G'(\rho)$ and (2.1)₂ by u and integrating over $\mathbb{R}^3 \times [0, T]$, then using (2.3), one gets

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \left(\frac{1}{2} \rho |u|^2 + G(\rho) \right) + \int_0^T \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2) \leq E_0, \quad (3.6)$$

which gives (3.4) and (3.5). \square

The following is the key *a priori* estimate which is essential to close the *a priori* assumptions (3.2).

Lemma 3.3 *Under the conditions of Proposition 3.1, assume further that $1 < \gamma \leq \frac{3}{2}$, we have*

$$\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{(\gamma-1)^{\frac{2}{3}} E_0^{\frac{2}{3}}}{\mu} E_1, \quad (3.7)$$

where $E_1 = C(\bar{\rho}, M) \left(1 + \frac{(\gamma-1)^{\frac{2}{9}}}{\mu^{\frac{2}{3}}} \right)$.

Proof. First, assume that $\frac{(\gamma-1)^{\frac{1}{3}} E_0}{\mu} \leq 1$. Multiplying (2.1)₂ by u and then integrating the resulting equality over \mathbb{R}^3 , and using integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2) = \int_{\mathbb{R}^3} P \operatorname{div} u. \quad (3.8)$$

Integrating (3.8) over $[0, \sigma(T)]$, using (3.4) and Cauchy inequality, we have

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} \frac{1}{2} \int_{\mathbb{R}^3} \rho |u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 \right) \\ & \leq \frac{1}{2} \int_{\mathbb{R}^3} \rho_0 |u_0|^2 + \frac{C}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |P|^2 \\ & \leq \frac{1}{2} \|\rho_0\|_{L^{\frac{3}{2}}} \|u_0\|_{L^6}^2 + \frac{C(\bar{\rho})(\gamma-1)E_0}{\mu} \\ & \leq C(\bar{\rho}, M)(\gamma-1)^{\frac{2}{3}} E_0^{\frac{2}{3}} \left(1 + \frac{(\gamma-1)^{\frac{1}{3}} E_0^{\frac{1}{3}}}{\mu} \right) \\ & \leq C(\bar{\rho}, M)(\gamma-1)^{\frac{2}{3}} E_0^{\frac{2}{3}} \left(1 + \frac{(\gamma-1)^{\frac{2}{9}}}{\mu^{\frac{2}{3}}} \right), \end{aligned} \quad (3.9)$$

where $1 < \gamma \leq \frac{3}{2}$ has been used. This completes the proof of Lemma 3.3. \square

Lemma 3.4 *Under the conditions of Proposition 3.1, it holds that*

$$\begin{aligned} A_1(T) & \leq \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|_{L^3}^3 + \frac{C\gamma}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma P |\nabla u|^2 \\ & \quad + \frac{C(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + \frac{C(\gamma-1)E_0}{\mu^2}, \end{aligned} \quad (3.10)$$

$$A_2(T) \leq C A_1(T) + \frac{C\gamma^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P \nabla u|^2 + C \left(\frac{2\mu + \lambda}{\mu} \right)^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4. \quad (3.11)$$

Proof. The proof of (3.10) and (3.11) is duo to Hoff [9] and Huang-Li-Xin [16]. For $m \geq 0$, multiplying (2.1)₂ by $\sigma^m \dot{u}$, integrating the resulting equality over \mathbb{R}^3 , we have

$$\int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 = \int_{\mathbb{R}^3} (-\sigma^m \dot{u} \cdot \nabla P + \mu \sigma^m \Delta u \cdot \dot{u} + (\lambda + \mu) \sigma^m \nabla \operatorname{div} u \cdot \dot{u})$$

$$= \sum_{i=1}^3 I_i. \quad (3.12)$$

Integrating by parts gives

$$\begin{aligned} I_1 &= - \int_{\mathbb{R}^3} \sigma^m \dot{u} \cdot \nabla P \\ &= \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u_t P + \int_{\mathbb{R}^3} \sigma^m \operatorname{div}(u \cdot \nabla u) P \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u P - m \int_{\mathbb{R}^3} \sigma^{m-1} \sigma' \operatorname{div} u P \\ &\quad - \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u P' \rho_t + \int_{\mathbb{R}^3} \sigma^m \operatorname{div}(u \cdot \nabla u) P \\ &\leq \left(\int_{\mathbb{R}^3} \sigma^m \operatorname{div} u P \right)_t - m \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \operatorname{div} u P \\ &\quad + (\gamma - 1) \sigma^m \int_{\mathbb{R}^3} P |\operatorname{div} u|^2 + C \int_{\mathbb{R}^3} \sigma^m P |\nabla u|^2, \end{aligned} \quad (3.13)$$

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^3} \mu \sigma^m \Delta u \cdot \dot{u} \\ &= - \int_{\mathbb{R}^3} \mu \sigma^m \nabla u \cdot \nabla u_t + \int_{\mathbb{R}^3} \mu \sigma^m \Delta u (u \cdot \nabla u) \\ &\leq -\frac{\mu}{2} \left(\int_{\mathbb{R}^3} \sigma^m |\nabla u|^2 \right)_t + \frac{\mu m}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |\nabla u|^2 + C \mu \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} I_3 &= \int_{\mathbb{R}^3} (\lambda + \mu) \sigma^m \nabla \operatorname{div} u \cdot \dot{u} \\ &\leq -\frac{\mu + \lambda}{2} \left(\sigma^m \int_{\mathbb{R}^3} |\operatorname{div} u|^2 \right)_t + \frac{m(\mu + \lambda)}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |\operatorname{div} u|^2 \\ &\quad + C(\mu + \lambda) \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3. \end{aligned} \quad (3.15)$$

Substituting (3.13)-(3.15) into (3.12) shows that

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\mu}{2} \sigma^m \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \sigma^m \|\operatorname{div} u\|_{L^2}^2 - \sigma^m \int_{\mathbb{R}^3} \operatorname{div} u P \right) + \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 \\ &\leq -m \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \operatorname{div} u P + C \gamma \int_{\mathbb{R}^3} \sigma^m P |\nabla u|^2 + C(2\mu + \lambda) \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 \\ &\quad + \frac{m(4\mu + 3\lambda)}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |\nabla u|^2. \end{aligned} \quad (3.16)$$

Applying (3.4), integrating (3.16) over $(0, T)$, we get

$$\sup_{0 \leq t \leq T} \left(\frac{\mu}{4} \sigma^m \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \sigma^m \|\operatorname{div} u\|_{L^2}^2 \right) + \int_0^T \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2$$

$$\begin{aligned}
&\leq \frac{C(\gamma-1)E_0}{\mu} + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |P| |\operatorname{div} u| + \frac{m(4\mu+3\lambda)}{2} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \\
&\quad + C(2\mu+\lambda) \int_0^T \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 + (3\gamma-2) \int_0^T \int_{\mathbb{R}^3} \sigma^m P |\nabla u|^2 \\
&\leq \frac{C(\gamma-1)E_0}{\mu} + C(2\mu+\lambda) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + C(2\mu+\lambda) \int_0^T \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 \\
&\quad + C\gamma \int_0^T \int_{\mathbb{R}^3} \sigma^m P |\nabla u|^2. \tag{3.17}
\end{aligned}$$

Choosing $m = 1$, one gets (3.10).

Next, for $m \geq 0$, operating $\sigma^m \dot{u}^j [\partial/\partial t + \operatorname{div}(u \cdot)]$ on (2.1)₂^j, summing over j , and integrating the resulting equation over $\mathbb{R}^3 \times [0, T]$, we obtain after integration by parts

$$\begin{aligned}
&\sup_{0 \leq t \leq T} \left(\frac{1}{2} \sigma^m \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right) - \frac{m}{2} \int_0^T \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \\
&= - \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\partial_j P_t + \operatorname{div}(\partial_j P u)] + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\Delta u_t^j + \operatorname{div}(u \Delta u^j)] \\
&\quad + (\lambda + \mu) \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\partial_t \partial_j \operatorname{div} u + \operatorname{div}(u \partial_j \operatorname{div} u)] \\
&= \sum_{i=1}^3 II_i. \tag{3.18}
\end{aligned}$$

Integrating by parts leads to

$$\begin{aligned}
II_1 &= - \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\partial_j P_t + \operatorname{div}(\partial_j P u)] \\
&= \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j P_t + \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j (\partial_j P u^k) \\
&= - \int_0^T \int_{\mathbb{R}^3} \sigma^m \operatorname{div} \dot{u} P' (\rho \operatorname{div} u + u \cdot \nabla \rho) + \int_0^T \int_{\mathbb{R}^3} \sigma^m \operatorname{div} \dot{u} \operatorname{div} u P \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \sigma^m \operatorname{div} \dot{u} u \cdot \nabla P - \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_j u^k P \\
&\leq C\gamma \int_0^T \int_{\mathbb{R}^3} \sigma^m P |\nabla \dot{u}| |\nabla u| \\
&\leq \frac{\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + \frac{C\gamma^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |P \nabla u|^2, \tag{3.19} \\
II_2 &= \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\Delta u_t^j + \operatorname{div}(u \Delta u^j)] \\
&= -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\partial_i \dot{u}^j|^2 + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_i \dot{u}^j \partial_i (u^k \partial_k u^j)
\end{aligned}$$

$$\begin{aligned}
& -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_i (u^k \partial_i u^j) - \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_i u^k \partial_i u^j \\
& \leq -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}| |\operatorname{div} u|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}| |\nabla u|^2 \\
& \leq -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}| |\nabla u|^2 \\
& \leq -\frac{3\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4.
\end{aligned} \tag{3.20}$$

Similarly,

$$\begin{aligned}
II_3 & \leq -\frac{\mu + \lambda}{2} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\operatorname{div} \dot{u}|^2 \\
& \quad + C(\mu + \lambda) \left(1 + \frac{\lambda}{\mu}\right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + \frac{\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2.
\end{aligned} \tag{3.21}$$

Substituting (3.19)-(3.21) into (3.18) shows that

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \sigma^m \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + (\mu + \lambda) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\operatorname{div} \dot{u}|^2 \\
& \leq C \int_0^T \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{C\gamma^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |P \nabla u|^2 \\
& \quad + \frac{C(2\mu + \lambda)^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4,
\end{aligned} \tag{3.22}$$

where (1.4) has been used. Taking $m = 2$, we immediately obtain (3.11). The proof of Lemma 3.4 is completed. \square

Lemma 3.5 *Under the conditions of Proposition 3.1, it holds that*

$$\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} \leq E_2, \tag{3.23}$$

$$\sup_{0 \leq t \leq \sigma(T)} t \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} t |\nabla \dot{u}|^2 \leq E_3, \tag{3.24}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ \left(4C(\bar{\rho})\right)^{-6}, 1 \right\} \triangleq \varepsilon_1$.

Proof. First, we assume that $\frac{(\gamma - 1)^{\frac{1}{3}} E_0}{\mu} \leq 1$. Multiplying (2.1)₂ by u_t , integrating the resulting equality over \mathbb{R}^3 and using (2.7), we have

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \|\operatorname{div} u\|_{L^2}^2 - \int_{\mathbb{R}^3} \operatorname{div} u P \right) + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \\
& = \int_{\mathbb{R}^3} \rho \dot{u} (u \cdot \nabla u) - \int_{\mathbb{R}^3} \operatorname{div} u P_t
\end{aligned}$$

$$\begin{aligned}
&\leq C(\bar{\rho}) \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \rho |u|^3 \right)^{\frac{1}{3}} \|\nabla u\|_{L^6} + \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div}(Pu) + (\gamma - 1) \int_{\mathbb{R}^3} P |\operatorname{div} u|^2 \\
&\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \rho |u|^3 \right)^{\frac{1}{3}} \left((CN_6 + 1) \|\rho \dot{u}\|_{L^2} + \|P\|_{L^6} \right) \\
&\quad - \int_{\mathbb{R}^3} Pu \cdot \nabla \operatorname{div} u + C(\bar{\rho})(\gamma - 1) \int_{\mathbb{R}^3} |\nabla u|^2 \\
&\leq \frac{C(\bar{\rho})\mu(CN_6 + 1)}{2\mu + \lambda} \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} A_3^{\frac{1}{3}}(\sigma(T)) + \frac{C(\bar{\rho})\mu}{2\mu + \lambda} \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} A_3^{\frac{1}{3}}(\sigma(T)) \|P\|_{L^6} \\
&\quad - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} Pu \cdot \nabla G + \frac{1}{2(2\mu + \lambda)} \int_{\mathbb{R}^3} \operatorname{div} u P^2 + C(\bar{\rho})(\gamma - 1) \|\nabla u\|_{L^2}^2 \\
&\leq C(\bar{\rho}) \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} A_3^{\frac{1}{3}}(\sigma(T)) + C(\bar{\rho}) A_3^{\frac{1}{3}}(\sigma(T)) \|P\|_{L^6}^2 + C(\bar{\rho})(\gamma - 1) \|\nabla u\|_{L^2}^2 \\
&\quad + \frac{C}{2\mu + \lambda} \|P\|_{L^3} \|\nabla u\|_{L^2} \|\rho \dot{u}\|_{L^2} + \frac{C}{2\mu + \lambda} (\|\nabla u\|_{L^2}^2 + \|P\|_{L^4}^4) \\
&\leq C(\bar{\rho}) \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} A_3^{\frac{1}{3}}(\sigma(T)) + C(\bar{\rho}) A_3^{\frac{1}{3}}(\sigma(T)) (\gamma - 1)^{\frac{1}{3}} E_0^{\frac{1}{3}} + C(\bar{\rho})(\gamma - 1) \|\nabla u\|_{L^2}^2 \\
&\quad + \frac{1}{4} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \frac{C(\bar{\rho})}{(2\mu + \lambda)^2} (\gamma - 1)^{\frac{2}{3}} E_0^{\frac{2}{3}} \|\nabla u\|_{L^2}^2 + \frac{C}{2\mu + \lambda} \|\nabla u\|_{L^2}^2 + \frac{C(\bar{\rho})}{2\mu + \lambda} (\gamma - 1) E_0 \\
&\leq C(\bar{\rho}) \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} A_3^{\frac{1}{3}}(\sigma(T)) + C(\bar{\rho}) A_3^{\frac{1}{3}}(\sigma(T)) (\gamma - 1)^{\frac{1}{3}} E_0^{\frac{1}{3}} + C(\bar{\rho})(\gamma - 1) \|\nabla u\|_{L^2}^2 \\
&\quad + \frac{1}{4} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \frac{C(\bar{\rho})}{(2\mu + \lambda)^2} (\gamma - 1)^{\frac{2}{3}} E_0^{\frac{2}{3}} \|\nabla u\|_{L^2}^2 + \frac{C}{2\mu + \lambda} \|\nabla u\|_{L^2}^2 \\
&\quad + \frac{C(\bar{\rho})}{2\mu + \lambda} (\gamma - 1) E_0, \tag{3.25}
\end{aligned}$$

where we have used (2.12). Integrating (3.25) over $(0, \sigma(T))$, we obtain that

$$\begin{aligned}
&\sup_{0 \leq t \leq \sigma(T)} \left\{ \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \|\operatorname{div} u\|_{L^2}^2 - \int_{\mathbb{R}^3} \operatorname{div} u P \right\} + \frac{1}{2} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \\
&\leq C(\bar{\rho}) A_3^{\frac{1}{3}}(\sigma(T)) (\gamma - 1)^{\frac{1}{3}} E_0^{\frac{1}{3}} + C(\bar{\rho})(\gamma - 1) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \\
&\quad + \frac{C(\bar{\rho})}{(2\mu + \lambda)^2} (\gamma - 1)^{\frac{2}{3}} E_0^{\frac{2}{3}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + \frac{C}{2\mu + \lambda} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \\
&\quad + \frac{C(\bar{\rho})}{2\mu + \lambda} (\gamma - 1) E_0 + C\mu M, \tag{3.26}
\end{aligned}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \left(4C(\bar{\rho}) \right)^{-6}$.

Case 1: $1 < \gamma \leq \frac{3}{2}$.

Using(3.7), we get

$$\sup_{0 \leq t \leq \sigma(T)} \{ \|\nabla u\|_{L^2}^2 + (\lambda + \mu) \|\operatorname{div} u\|_{L^2}^2 \} + \frac{1}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \leq E_2^1, \quad (3.27)$$

where

$$E_2^1 = \frac{C(\gamma - 1)^{\frac{2}{9}}}{\mu^{\frac{2}{3}}} + \frac{C(\gamma - 1)^{\frac{13}{9}} E_1}{\mu^{\frac{4}{3}}} + \frac{C(\gamma - 1)^{\frac{8}{9}} E_1}{\mu^{\frac{8}{3}}} + \frac{C(\gamma - 1)^{\frac{4}{9}} E_1}{\mu^{\frac{7}{3}}} + \frac{C(\gamma - 1)^{\frac{2}{3}}}{\mu} + CM,$$

and we have also used the facts that $\frac{(\gamma - 1)^{\frac{1}{3}} E_0}{\mu} \leq 1$ and $\mu + \lambda > 0$.

Case 2: $\gamma > \frac{3}{2}$.

Using(3.5), we have

$$\sup_{0 \leq t \leq \sigma(T)} \{ \|\nabla u\|_{L^2}^2 + (\lambda + \mu) \|\operatorname{div} u\|_{L^2}^2 \} + \frac{1}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \leq E_2^2, \quad (3.28)$$

where

$$E_2^2 = \frac{C(\gamma - 1)^{\frac{2}{9}}}{\mu^{\frac{2}{3}}} + \frac{C(\gamma - 1)^{\frac{2}{3}}}{\mu} + \frac{C(\gamma - 1)^{\frac{1}{9}}}{\mu^{\frac{7}{3}}} + \frac{C}{\mu^2} + \frac{C(\gamma - 1)^{\frac{2}{3}}}{\mu} + CM.$$

Combining (3.27)-(3.28), the result leads to (3.23), where

$$E_2 = \max \{ E_2^1, E_2^2 \}. \quad (3.29)$$

Taking $m = 1$ and $T = \sigma(T)$ in (3.22), we obtain that

$$\begin{aligned} & \sigma \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 + (\mu + \lambda) \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\operatorname{div} \dot{u}|^2 \\ & \leq \mu E_2 + \frac{C\gamma^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P \nabla u|^2 + \frac{C(2\mu + \lambda)^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \\ & \leq \mu E_2 + \frac{C\gamma^2}{\mu} \left(\int_0^{\sigma(T)} \int_{\mathbb{R}^3} P^4 \right)^{\frac{1}{2}} \left(\int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{2}} \\ & \quad + \frac{C(2\mu + \lambda)^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \\ & \leq \mu E_2 + \frac{C\gamma^2(\gamma - 1)^{\frac{1}{3}} E_2^{\frac{3}{4}}}{\mu^{\frac{5}{4}}} + \frac{C\gamma^2(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{1}{4}}}{\mu^{\frac{7}{4}}} + C \left(2 + \frac{\lambda}{\mu} \right)^2 \left(\frac{E_2^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} + \frac{(\gamma - 1)^{\frac{1}{3}} E_2^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \right) \\ & \leq \mu E_3, \end{aligned} \quad (3.30)$$

where

$$E_3 = E_2 + \frac{C\gamma^2(\gamma - 1)^{\frac{1}{3}} E_2^{\frac{3}{4}}}{\mu^{\frac{9}{4}}} + \frac{C\gamma^2(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{1}{4}}}{\mu^{\frac{11}{4}}} + C \left(2 + \frac{\lambda}{\mu} \right)^2 \left(\frac{E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{(\gamma - 1)^{\frac{1}{3}} E_2^{\frac{1}{2}}}{\mu^{\frac{5}{2}}} \right).$$

To get (3.30), we have used the following estimate

$$\begin{aligned}
& \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \\
& \leq \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2} \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^6}^3 \\
& \leq E_2^{\frac{1}{2}} \int_0^{\sigma(T)} \sigma \left(\frac{(4\mu + \lambda)^3}{(2\mu + \lambda)^3 \mu^3} \|\rho \dot{u}\|_{L^2}^3 + \frac{\|P\|_{L^6}^3}{(2\mu + \lambda)^3} \right) \\
& \leq \frac{CE_2^{\frac{1}{2}}}{\mu^3} \sup_{0 \leq t \leq \sigma(T)} \sigma \|\rho \dot{u}\|_{L^2} \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 + \frac{CE_2^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^3} \\
& \leq \frac{CE_2^{\frac{1}{2}}}{\mu^3} (\mu A_2(T))^{\frac{1}{2}} \mu E_2 + \frac{CE_2^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{3}}}{\mu^{\frac{5}{2}}} \\
& \leq \frac{CE_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{CE_2^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{3}}}{\mu^{\frac{5}{2}}}, \tag{3.31}
\end{aligned}$$

due to Hölder inequality, (2.12), (3.2), the facts that $\frac{(\gamma - 1)^{\frac{1}{3}} E_0}{\mu} \leq 1$ and $\mu + \lambda > 0$. The proof of Lemma 3.5 is completed. \square

Lemma 3.6 *Under the conditions of Proposition 3.1, we have*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} \leq C(E_2 + 1), \tag{3.32}$$

$$\sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^T \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \leq C(E_3 + 1), \tag{3.33}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \varepsilon_1$.

Proof. By (3.2) and Lemma 3.5, we immediately get Lemma 3.6. \square

Next, we will close the *a priori* assumption on $A_3(T)$.

Lemma 3.7 *Under the conditions of Proposition 3.1, it holds that*

$$A_3(\sigma(T)) \leq \left\{ \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right\}^{\frac{1}{2}}, \tag{3.34}$$

provided

$$\begin{aligned}
\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} & \leq \min \left\{ C(\bar{\rho})^{-2} (\gamma - 1)^{-\frac{2}{3}} E_2^{-3} \mu^5 \Big|_{(1 < \gamma \leq \frac{3}{2})}, C(\bar{\rho})^{-1} \mu^{\frac{9}{4}} E_2^{-\frac{3}{4}} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_1 \right\} \\
& \triangleq \varepsilon_2. \tag{3.35}
\end{aligned}$$

Proof. Case 1: $1 < \gamma \leq \frac{3}{2}$.

$$\begin{aligned}
A_3(\sigma(T)) &= \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \frac{\rho|u|^3}{\mu^3} \leq \frac{1}{\mu^3} \sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^2} \|u\|_{L^6}^3 \\
&\leq \frac{C(\bar{\rho})}{\mu^3} (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} E_2^{\frac{3}{2}} \\
&\leq \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{1}{2}}, \tag{3.36}
\end{aligned}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq C(\bar{\rho})(\gamma - 1)^{-\frac{2}{3}} E_2^{-3} \mu^5$, where we have used Hölder inequality, Sobolev inequality and Lemma 3.5.

Case 2: $\gamma > \frac{3}{2}$.

Using Hölder inequality, (3.6), (3.23) and $2(\gamma - 1) \geq 1$, we obtain

$$\begin{aligned}
A_3(\sigma(T)) &= \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \frac{\rho|u|^3}{\mu^3} \leq \frac{C(\bar{\rho})}{\mu^3} \sup_{0 \leq t \leq \sigma(T)} \|\rho^{\frac{1}{2}} u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^6}^{\frac{3}{2}} \\
&\leq \frac{C(\bar{\rho})(\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_2^{\frac{3}{4}}}{\mu^3} \\
&\leq \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{1}{2}}, \tag{3.37}
\end{aligned}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq C(\bar{\rho}) \mu^{\frac{9}{4}} E_2^{-\frac{3}{4}}$. □

Lemma 3.8 *Under the conditions of Proposition 3.1, it holds that*

$$A_1(T) + A_2(T) \leq \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \tag{3.38}$$

provided

$$\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ (CE_7)^{-3} \Big|_{(1 < \gamma \leq \frac{3}{2})}, (CE_{11})^{-2} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_2 \right\} \triangleq \varepsilon_3,$$

where E_7 and E_{11} will be determined in (3.53) and (3.65).

Proof. First we assume that $\frac{(\gamma - 1)^{\frac{1}{3}} E_0}{\mu} \leq 1$. It follows from Lemma 3.4 that

$$\begin{aligned}
&A_1(T) + A_2(T) \\
&\leq \frac{C(\gamma - 1)E_0}{\mu^2} + \frac{C(2\mu + \lambda)^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 + \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|_{L^3}^3
\end{aligned}$$

$$\begin{aligned}
& + \frac{C\gamma^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P \nabla u|^2 + \frac{C\gamma}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma P |\nabla u|^2 + C \left(\frac{2\mu + \lambda}{\mu} \right) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \\
& \leq \frac{C(\gamma - 1)E_0}{\mu^2} + \sum_{i=4}^8 II_i.
\end{aligned} \tag{3.39}$$

Now we give the estimates of $II_4 - II_8$ in the following two cases. The subsequent estimates proceed with different techniques for $1 < \gamma \leq \frac{3}{2}$ and $\gamma > \frac{3}{2}$.

Case 1: $1 < \gamma \leq \frac{3}{2}$.

For II_4 , due to (2.12), we just estimate $\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4$ as follows:

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 & \leq CN_4^4 (2\mu + \lambda)^{-3} \int_0^T \sigma^2 \|\rho \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2} \\
& \quad + C(2\mu + \lambda)^{-4} \int_0^T \sigma^2 \|\rho \dot{u}\|_{L^2}^3 \|P\|_{L^2} \\
& \quad + C(2\mu + \lambda)^{-4} \int_0^T \sigma^2 \|P\|_{L^4}^4 \\
& = \sum_{i=1}^3 III_i.
\end{aligned} \tag{3.40}$$

Using Hölder inequality, (3.2), (3.4) and Lemma 3.6, we have

$$\begin{aligned}
III_1 & \leq CN_4^4 (2\mu + \lambda)^{-3} \sup_{0 \leq t \leq T} (\sigma^2 \|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|\nabla u\|_{L^2} \\
& \leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{1}{2}} A_2^{\frac{1}{2}}(T) \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho\|_{L^3}^2 \|\nabla \dot{u}\|_{L^2}^2 \\
& \leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{1}{2}} A_2^{\frac{1}{2}}(T) (\gamma - 1)^{\frac{2}{3}} E_0^{\frac{2}{3}} (E_2 + 1)^{\frac{1}{2}} (E_3 + 1) \\
& \leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{1}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} (E_2 + 1)^{\frac{1}{2}} (E_3 + 1).
\end{aligned} \tag{3.41}$$

Next, it follows from Hölder inequality, (3.2), (3.4) and Lemma 3.6 that

$$\begin{aligned}
III_2 & \leq C(2\mu + \lambda)^{-4} \sup_{0 \leq t \leq T} (\sigma^2 \|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|P\|_{L^2} \\
& \leq C(2\mu + \lambda)^{-4} \mu^{\frac{1}{2}} A_2^{\frac{1}{2}}(T) \sup_{0 \leq t \leq T} \|P\|_{L^2} \int_0^T \sigma \|\rho\|_{L^3}^2 \|\nabla \dot{u}\|_{L^2}^2 \\
& \leq C(2\mu + \lambda)^{-4} \mu^{\frac{1}{4}} (\gamma - 1)^{\frac{5}{4}} E_0^{\frac{17}{12}} (E_3 + 1).
\end{aligned} \tag{3.42}$$

Thus, one gets from (2.8), (3.41) and (3.42) that

$$\int_0^T \sigma^2 \|G\|_{L^4}^4 \leq C \left((2\mu + \lambda) \|\nabla u\|_{L^2} + \|P\|_{L^2} \right) \|\rho \dot{u}\|_{L^2}^3$$

$$\begin{aligned}
&\leq C(2\mu + \lambda)\mu^{\frac{1}{4}}(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{12}}\left(E_2 + 1\right)^{\frac{1}{2}}\left(E_3 + 1\right) \\
&\quad + C\mu^{\frac{1}{4}}(\gamma - 1)^{\frac{5}{4}}E_0^{\frac{17}{12}}\left(E_3 + 1\right) \\
&\leq (2\mu + \lambda)\mu^{\frac{1}{4}}(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{12}}E_4,
\end{aligned} \tag{3.43}$$

where $E_4 = C\left(E_2 + 1\right)^{\frac{1}{2}}\left(E_3 + 1\right) + C(\gamma - 1)^{\frac{1}{3}}\mu^{-\frac{1}{2}}\left(E_3 + 1\right)$. Here we have used the facts that $\frac{(\gamma - 1)^{\frac{1}{3}}E_0}{\mu} \leq 1$ and $\mu + \lambda > 0$.

To estimate III_3 , one deduces from (2.1)₁ that P satisfies

$$P_t + u \cdot \nabla P + \gamma P \operatorname{div} u = 0. \tag{3.44}$$

Multiplying (3.44) by $3\sigma^2 P^2$ and integrating the resulting equality over $\mathbb{R}^3 \times [0, T]$, one gets that

$$\begin{aligned}
&\frac{3\gamma - 1}{2\mu + \lambda} \int_0^T \sigma^2 \|P\|_{L^4}^4 \\
&= -\sigma^2 \|P\|_{L^3}^3 + 2\sigma\sigma' \int_0^T \|P\|_{L^3}^3 - \frac{3\gamma - 1}{2\mu + \lambda} \int_0^T \sigma^2 \int_{\mathbb{R}^3} P^3 G \\
&\leq C\|P\|_{L^3}^3 + \frac{3\gamma - 1}{4\mu + 2\lambda} \int_0^T \sigma^2 \|P\|_{L^4}^4 + \frac{C(3\gamma - 1)}{2\mu + \lambda} \int_0^T \sigma^2 \|G\|_{L^4}^4.
\end{aligned} \tag{3.45}$$

The combination of (3.43) with (3.45) implies

$$\begin{aligned}
\int_0^T \sigma^2 \|P\|_{L^4}^4 &\leq C(2\mu + \lambda)(\gamma - 1)E_0 + C(2\mu + \lambda)\mu^{\frac{1}{4}}(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{12}}E_4 \\
&\leq (2\mu + \lambda)(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{12}}\mu^{\frac{1}{4}}E_5,
\end{aligned} \tag{3.46}$$

where $E_5 = C(\gamma - 1)^{\frac{2}{9}}\mu^{-\frac{1}{6}} + CE_4$.

Substituting (3.41), (3.42) and (3.46) into (3.40) shows that

$$\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq (2\mu + \lambda)^{-3} \mu^{\frac{1}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6, \tag{3.47}$$

where $E_6 = CN_4^4 E_4 + CE_5$. Thus,

$$II_4 = C \left(\frac{2\mu + \lambda}{\mu} \right)^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq \frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6}{(2\mu + \lambda) \mu^{\frac{7}{4}}}. \tag{3.48}$$

To estimate II_5 , using Hölder inequality, (3.5) and (3.47), we have

$$\begin{aligned}
II_5 &= \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|_{L^3}^3 \\
&\leq \frac{C(2\mu + \lambda)}{\mu} \left(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C \frac{(\gamma-1)^{\frac{3}{8}} E_0^{\frac{23}{24}} E_6^{\frac{1}{2}}}{\mu^{\frac{11}{8}} (2\mu + \lambda)^{\frac{1}{2}}}. \quad (3.49)$$

For II_6 , using Hölder inequality, (3.46) and (3.47), one gets

$$\begin{aligned} II_6 &= \frac{C\gamma^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P \nabla u|^2 \\ &\leq \frac{C\gamma^2}{\mu^2} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |P|^4 \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{2}} \\ &\leq \frac{C\gamma^2 E_5^{\frac{1}{2}} E_6^{\frac{1}{2}} (\gamma-1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{7}{4}} (2\mu + \lambda)}. \end{aligned} \quad (3.50)$$

It holds from Hölder inequality, (3.5) and (3.46)-(3.47) that

$$\begin{aligned} II_7 &= \frac{C\gamma}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma P |\nabla u|^2 \\ &\leq \frac{C\gamma}{\mu} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |P|^4 \right)^{\frac{1}{4}} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{4}} \left(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C\gamma E_5^{\frac{1}{4}} E_6^{\frac{1}{4}} (\gamma-1)^{\frac{3}{8}} E_0^{\frac{23}{24}}}{\mu^{\frac{11}{8}} (2\mu + \lambda)^{\frac{1}{2}}}. \end{aligned} \quad (3.51)$$

Finally, relations (3.7), (3.39) and (3.48)-(3.51) give rise to

$$\begin{aligned} A_1(T) + A_2(T) &\leq \frac{C(\gamma-1)E_0}{\mu^2} + \frac{C(\gamma-1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6}{(2\mu + \lambda)\mu^{\frac{7}{4}}} + \frac{C(\gamma-1)^{\frac{3}{8}} E_0^{\frac{23}{24}} E_6^{\frac{1}{2}}}{\mu^{\frac{11}{8}} (2\mu + \lambda)^{\frac{1}{2}}} \\ &\quad + \frac{C\gamma^2 E_5^{\frac{1}{2}} E_6^{\frac{1}{2}} (\gamma-1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{7}{4}} (2\mu + \lambda)} + \frac{C\gamma E_5^{\frac{1}{4}} E_6^{\frac{1}{4}} (\gamma-1)^{\frac{3}{8}} E_0^{\frac{23}{24}}}{\mu^{\frac{11}{8}} (2\mu + \lambda)^{\frac{1}{2}}} \\ &\quad + \frac{C(2\mu + \lambda)(\gamma-1)^{\frac{2}{3}} E_0^{\frac{2}{3}} E_1}{\mu^2} \\ &\leq C \left(\frac{(\gamma-1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{4}{3}} E_7, \end{aligned} \quad (3.52)$$

where

$$\begin{aligned} E_7 &= \frac{(\gamma-1)^{\frac{2}{3}}}{\mu} + \frac{(\gamma-1)^{\frac{4}{9}} E_6}{\mu^{\frac{11}{6}}} + \frac{(\gamma-1)^{\frac{1}{18}} E_6^{\frac{1}{2}}}{\mu^{\frac{11}{12}}} + \frac{(\gamma-1)^{\frac{4}{9}} E_5^{\frac{1}{2}} E_6^{\frac{1}{2}}}{\mu^{\frac{11}{6}}} \\ &\quad + \frac{\gamma(\gamma-1)^{\frac{1}{18}} E_5^{\frac{1}{4}} E_6^{\frac{1}{4}}}{\mu^{\frac{11}{12}}} + \left(2 + \frac{\lambda}{\mu} \right) \frac{(\gamma-1)^{\frac{4}{9}} E_1}{\mu^{\frac{1}{3}}}, \end{aligned} \quad (3.53)$$

and we have also used the facts that $\frac{(\gamma-1)^{\frac{1}{3}}E_0}{\mu} \leq 1$ and $\mu + \lambda > 0$. It thus follows from (3.52) that

$$A_1(T) + A_2(T) \leq C \left(\frac{(\gamma-1)^{\frac{1}{6}}E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{4}{3}} E_7 \leq \frac{(\gamma-1)^{\frac{1}{6}}E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \quad (3.54)$$

provided $\frac{(\gamma-1)^{\frac{1}{6}}E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq (CE_7)^{-3}$.

Case 2: $\gamma > \frac{3}{2}$.

In view of (3.40), one gets

$$\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq \sum_{i=1}^3 III_i. \quad (3.55)$$

For III_1 , using (3.2) and Lemma 3.6, we have

$$\begin{aligned} III_1 &\leq CN_4^4 (2\mu + \lambda)^{-3} \sup_{0 \leq t \leq T} (\sigma^2 \|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|\nabla u\|_{L^2} \\ &\leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{1}{2}} A_2^{\frac{1}{2}}(T) \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \\ &\leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{3}{2}} A_2^{\frac{1}{2}}(T) A_1(T) (E_2 + 1)^{\frac{1}{2}} \\ &\leq CN_4^4 (2\mu + \lambda)^{-3} \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} (E_2 + 1)^{\frac{1}{2}}. \end{aligned} \quad (3.56)$$

It follows from (3.2), (3.4) and Lemma 3.6 that

$$\begin{aligned} III_2 &\leq C(2\mu + \lambda)^{-4} \sup_{0 \leq t \leq T} (\sigma^2 \|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|P\|_{L^2} \\ &\leq C(2\mu + \lambda)^{-4} \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{5}{4}} (E_2 + 1)^{\frac{1}{2}}. \end{aligned} \quad (3.57)$$

Thus, one gets from (2.8), (3.56) and (3.57) that

$$\int_0^T \sigma^2 \|G\|_{L^4}^4 \leq (2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_8, \quad (3.58)$$

where

$$E_8 = C(E_2 + 1)^{\frac{1}{2}} \left(1 + \frac{(\gamma - 1)^{\frac{1}{3}}}{\mu^{\frac{1}{2}}} \right). \quad (3.59)$$

Using (3.45), $\frac{1}{\gamma - 1} < 2$ and $\frac{(\gamma - 1)^{\frac{1}{3}}E_0}{\mu} \leq 1$, we obtain

$$\int_0^T \sigma^2 \|P\|_{L^4}^4 \leq C(2\mu + \lambda)(\gamma - 1)E_0 + C(3\gamma - 1)(2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_8$$

$$\begin{aligned}
&\leq C(2\mu + \lambda)\mu^{\frac{3}{4}}(\gamma - 1)E_0^{\frac{3}{4}} \left[E_0^{\frac{1}{4}}\mu^{-\frac{3}{4}} + (3\gamma - 1)(\gamma - 1)^{-\frac{3}{4}}E_8 \right] \\
&\leq C(2\mu + \lambda)\mu^{\frac{3}{4}}(\gamma - 1)E_0^{\frac{3}{4}} \left[E_0^{\frac{1}{4}}\mu^{-\frac{3}{4}} + 2^{\frac{3}{4}}(3\gamma - 1)E_8 \right] \\
&\leq (2\mu + \lambda)\mu^{\frac{3}{4}}(\gamma - 1)E_0^{\frac{3}{4}}E_9,
\end{aligned} \tag{3.60}$$

where $E_9 = C\mu^{-\frac{1}{2}} + C(3\gamma - 1)E_8$.

Substituting (3.56), (3.57) and (3.60) into (3.55), we get

$$\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq (2\mu + \lambda)^{-3} \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_{10}, \tag{3.61}$$

where $E_{10} = CN_4^4(E_2 + 1)^{\frac{1}{2}} + C\mu^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{3}}(E_2 + 1)^{\frac{1}{2}} + C(\gamma - 1)^{\frac{3}{4}}E_9$. Thus,

$$II_4 \leq C \left(\frac{2\mu + \lambda}{\mu} \right)^2 (2\mu + \lambda)^{-3} \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_{10} \leq \frac{C(\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_{10}}{\mu^{\frac{5}{4}}(2\mu + \lambda)}. \tag{3.62}$$

Using the same spirit of deriving (3.49)-(3.51), we obtain

$$\left\{ \begin{aligned} II_5 &\leq \frac{C(2\mu + \lambda)}{\mu} \left(\frac{E_0}{\mu} \right)^{\frac{1}{2}} \left((2\mu + \lambda)^{-3} \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{3}{4}} E_{10} \right)^{\frac{1}{2}}, \\ &\leq \frac{C(\gamma - 1)^{\frac{1}{8}} E_0^{\frac{7}{8}} E_{10}^{\frac{1}{2}}}{\mu^{\frac{9}{8}}(2\mu + \lambda)^{\frac{1}{2}}}, \\ II_6 &\leq \frac{C\gamma^2(\gamma - 1)^{\frac{5}{8}} E_0^{\frac{3}{4}} E_9^{\frac{1}{2}} E_{10}^{\frac{1}{2}}}{\mu^{\frac{5}{4}}(2\mu + \lambda)}, \\ II_7 &\leq \frac{C\gamma(\gamma - 1)^{\frac{5}{16}} E_0^{\frac{7}{8}} E_9^{\frac{1}{4}} E_{10}^{\frac{1}{4}}}{\mu^{\frac{9}{8}}(2\mu + \lambda)^{\frac{1}{2}}}, \\ II_8 &\leq \frac{C(2\mu + \lambda)E_0}{\mu^2} \leq \frac{C(2\mu + \lambda)(\gamma - 1)E_0}{\mu^2}. \end{aligned} \right. \tag{3.63}$$

Consequently, after a bit tedious but straightforward calculation, relations (3.7), (3.39) and (3.63) give rise to

$$\begin{aligned}
A_1(T) + A_2(T) &\leq \frac{C(\gamma - 1)E_0}{\mu^2} + \frac{C(\gamma - 1)^{\frac{1}{4}} E_0^{\frac{5}{4}} E_{10}}{\mu^{\frac{3}{4}}(2\mu + \lambda)} + \frac{C(\gamma - 1)^{\frac{1}{8}} E_0^{\frac{7}{8}} E_{10}^{\frac{1}{2}}}{\mu^{\frac{9}{8}}(2\mu + \lambda)^{\frac{1}{2}}} \\
&\quad + \frac{C\gamma^2(\gamma - 1)^{\frac{5}{8}} E_0^{\frac{3}{4}} E_9^{\frac{1}{2}} E_{10}^{\frac{1}{2}}}{\mu^{\frac{5}{4}}(2\mu + \lambda)} + \frac{C\gamma(\gamma - 1)^{\frac{5}{16}} E_0^{\frac{7}{8}} E_9^{\frac{1}{4}} E_{10}^{\frac{1}{4}}}{\mu^{\frac{9}{8}}(2\mu + \lambda)^{\frac{1}{2}}} \\
&\quad + \frac{C(2\mu + \lambda)(\gamma - 1)E_0}{\mu^2} \\
&\leq C \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{2}} E_{11},
\end{aligned} \tag{3.64}$$

where

$$E_{11} = \frac{(\gamma - 1)^{\frac{2}{3}}}{\mu} + \frac{E_{10}}{\mu} + \frac{E_{10}^{\frac{1}{2}}}{\mu^{\frac{3}{4}}} + \frac{\gamma^2(\gamma - 1)^{\frac{3}{8}} E_9^{\frac{1}{2}} E_{10}^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} + \frac{\gamma(\gamma - 1)^{\frac{1}{48}} E_9^{\frac{1}{4}} E_{10}^{\frac{1}{4}}}{\mu^{\frac{3}{4}}}$$

$$+ \left(2 + \frac{\lambda}{\mu}\right) (\gamma - 1)^{\frac{2}{3}}. \quad (3.65)$$

Here we have used the facts that $\frac{1}{\gamma - 1} < 2$, $\frac{(\gamma - 1)^{\frac{1}{3}} E_0}{\mu} \leq 1$ and $\mu + \lambda > 0$. It thus follows from (3.64) that

$$A_1(T) + A_2(T) \leq C \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{2}} E_{11} \leq \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \quad (3.66)$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq (CE_{11})^{-2}$.

By (3.54) and (3.66), for **Case 1** and **Case 2**, we conclude that if

$$\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ (CE_7)^{-3} \Big|_{(1 < \gamma \leq \frac{3}{2})}, (CE_{11})^{-2} \Big|_{(\gamma > \frac{3}{2})}, \varepsilon_1, \varepsilon_2 \right\} \triangleq \varepsilon_3,$$

then

$$A_1(T) + A_2(T) \leq \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}. \quad (3.67)$$

The proof of Lemma 3.8 is completed. \square

Next, we will derive the time-independent upper bound for the density ρ . The approach is motivated by Huang-Li-Xin in [16] and Li-Xin in [22].

Lemma 3.9 *Under the conditions of Proposition 3.1, it holds that*

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \frac{7\bar{\rho}}{4} \quad (3.68)$$

for any $(x, t) \in \mathbb{R}^3 \times [0, T]$, provided

$$\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ \varepsilon_3, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\} \triangleq \varepsilon.$$

Proof. Denoting $D_t \rho = \rho_t + u \cdot \nabla \rho$ and expressing the equation of the mass conservation (1.1)₁ as

$$D_t \rho = g(\rho) + b'(t),$$

where

$$g(\rho) \triangleq -\frac{\rho P}{2\mu + \lambda}, \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_0^t \rho G d\tau.$$

In accordance with Lemma 2.1, (2.7), (2.8), (3.1) and Lemma 3.8, for all $0 \leq t_1 < t_2 \leq \sigma(T)$, we get

$$|b(t_2) - b(t_1)| \leq C \int_0^{\sigma(T)} \|(\rho G)(\cdot, t)\|_{L^\infty} dt$$

$$\begin{aligned}
&\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \int_0^{\sigma(T)} \|G(\cdot, t)\|_{L^6}^{\frac{1}{2}} \|\nabla G(\cdot, t)\|_{L^6}^{\frac{1}{2}} dt \\
&\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \dot{u}\|_{L^2}^{\frac{1}{2}} dt \\
&\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \left(\int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^{\frac{2}{3}} t^{-\frac{1}{3}} dt \right)^{\frac{3}{4}} \left(\int_0^{\sigma(T)} t \|\nabla \dot{u}\|_{L^2}^2 dt \right)^{\frac{1}{4}} \\
&\leq \frac{C(\bar{\rho}, M)}{2\mu + \lambda} \left(\int_0^{\sigma(T)} (\|\rho \dot{u}\|_{L^2}^2 t)^{\frac{1}{3}} t^{-\frac{2}{3}} dt \right)^{\frac{3}{4}} \\
&\leq \frac{C(\bar{\rho}, M)}{2\mu + \lambda} \sup_{0 \leq t \leq \sigma(T)} (\|\rho \dot{u}\|_{L^2}^2 t)^{\frac{1}{16}} \left(\int_0^{\sigma(T)} (\|\rho \dot{u}\|_{L^2}^2 t)^{\frac{1}{4}} t^{-\frac{2}{3}} dt \right)^{\frac{3}{4}} \\
&\leq \frac{C(\bar{\rho}, M) \mu^{\frac{1}{16}}}{2\mu + \lambda} \left(\int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t \right)^{\frac{3}{16}} \left(\int_0^{\sigma(T)} t^{-\frac{8}{9}} dt \right)^{\frac{9}{16}} \\
&\leq \frac{C(\bar{\rho}, M)}{\mu^{\frac{3}{4}}} A_1(\sigma(T))^{\frac{3}{16}} \\
&\leq \frac{C(\bar{\rho}, M)}{\mu^{\frac{3}{4}}} \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{16}}, \tag{3.69}
\end{aligned}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \varepsilon_3$. Therefore, for $t \in [0, \sigma(T)]$, one can choose N_0 and N_1 in Lemma 2.3 as follows:

$$N_1 = 0, \quad N_0 = \frac{C(\bar{\rho}, M)}{\mu^{\frac{3}{4}}} \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{16}},$$

and $\bar{\zeta} = 0$. Then

$$g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = 0 \quad \text{for all } \zeta \geq \bar{\zeta} = 0.$$

Thus

$$\sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^\infty} \leq \max\{\bar{\rho}, 0\} + N_0 \leq \bar{\rho} + \frac{C(\bar{\rho}, M)}{\mu^{\frac{3}{4}}} \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{16}} \leq \frac{3\bar{\rho}}{2}, \tag{3.70}$$

provided

$$\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min\left\{\varepsilon_3, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4\right\}. \tag{3.71}$$

On the other hand, by virtues of Lemma 2.1, (2.7), (3.1) and Lemma 3.8, for $t \in [\sigma(T), T]$, one deduces that

$$|b(t_2) - b(t_1)| \leq \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{t_1}^{t_2} \|G(\cdot, t)\|_{L^\infty} dt$$

$$\begin{aligned}
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)}^T \|G\|_{L^\infty}^4 \\
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)}^T \|G\|_{L^6}^2 \|\nabla G\|_{L^6}^2 \\
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)}^T \|\nabla G\|_{L^2}^2 \|\nabla \dot{u}\|_{L^2}^2 \\
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)}^T \|\rho \dot{u}\|_{L^2}^2 \|\nabla \dot{u}\|_{L^2}^2 \\
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + C(\bar{\rho}) A_2(T) \int_{\sigma(T)}^T \|\nabla \dot{u}\|_{L^2}^2 \\
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + C(\bar{\rho}) A_2^2(T) \\
&\leq \frac{1}{2\mu + \lambda}(t_2 - t_1) + C(\bar{\rho}) \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^2, \tag{3.72}
\end{aligned}$$

provided $\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \varepsilon_3$.

Therefore, one can choose N_1 and N_0 in Lemma 2.3 as

$$N_1 = \frac{1}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho}) \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^2.$$

Note that

$$g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = -\frac{1}{2\mu + \lambda} \quad \text{for all } \zeta \geq 1,$$

one can set $\bar{\zeta} = 1$. Thus

$$\begin{aligned}
\sup_{\sigma(T) \leq s \leq T} \|\rho\|_{L^\infty} &\leq \max \left\{ \frac{3}{2} \bar{\rho}, 1 \right\} + N_0 \leq \frac{3}{2} \bar{\rho} + C(\bar{\rho}) \left(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^2 \\
&\leq \frac{7\bar{\rho}}{4}, \tag{3.73}
\end{aligned}$$

provided

$$\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ \varepsilon_3, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}. \tag{3.74}$$

The combination of (3.70) and (3.73) completes the proof of Lemma 3.9. \square

Proof of Theorem 1.1: With Lemma 3.9 and the higher norm estimates of the smooth solution (ρ, u) in [16], Theorem 1.1 follows.

4 The proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Let

$$\begin{cases} A_1(T) = \sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \sigma \frac{\rho |\dot{u}|^2}{\mu}, \\ A_2(T) = \sup_{0 \leq t \leq T} \sigma^3 \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2, \\ A_3(T) = \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \frac{\rho |u|^3}{\mu^3}. \end{cases} \quad (4.1)$$

Throughout the rest of the paper, we denote generic constant by C depending on $\bar{\rho}, M$ and some other known constants but independent of $\mu, \lambda, \gamma - 1, \tilde{\rho}$ and t , and we write $C(\alpha)$ to emphasize that C may depend on α . For simplicity of presentation, we shall assume that

$$\frac{(\gamma - 1 + \tilde{\rho}) E_0^\alpha}{\mu^\beta} \leq 1, \quad (4.2)$$

where $\frac{1}{10} \leq \alpha \leq 100$, $\frac{2}{3} \leq \beta \leq 12$. It's worth noting that the assumption (4.2) is not essential for our paper. Without this assumption, $E_{12}-E_{21}$ in the proof of Theorem 1.2 should be more complex. The following proposition plays a crucial role in this section.

Proposition 4.1 *Assume that the initial data satisfies (1.11), (1.12) and (1.13), $1 < \gamma < 2$. If the solution (ρ, u) satisfies*

$$\begin{cases} 0 \leq \rho \leq 2\bar{\rho}, & A_1(T) \leq 2 \left\{ \frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right\}^{\frac{3}{8}}, \\ A_2(T) \leq \frac{2 \left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}}, & A_3(\sigma(T)) \leq 2 \left\{ \frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right\}^{\frac{1}{2}}, \end{cases} \quad (4.3)$$

then

$$\begin{cases} 0 \leq \rho \leq \frac{7}{4}\bar{\rho}, & A_1(T) \leq \left\{ \frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right\}^{\frac{3}{8}}, \\ A_2(T) \leq \frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}}, & A_3(\sigma(T)) \leq \left\{ \frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right\}^{\frac{1}{2}}, \end{cases} \quad (4.4)$$

$(x, t) \in \mathbb{R}^3 \times [0, T]$, provided $\frac{(\gamma - 1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \frac{\tilde{\rho}}{2C}$ and $\frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \varepsilon$. Here

$$\varepsilon = \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\},$$

and

$$\begin{aligned}\varepsilon_6 &= \min \left\{ \left(C(E_{18} + E_{19} + E_{20}) \right)^{-17}, \left(C(E_{18} + E_{19} + E_{21}) \right)^{-8}, \varepsilon_5 \right\}, \\ \varepsilon_5 &= \min \left\{ \left(C(E_{15}E_{17} + E_{16}) \right)^{-4}, \varepsilon_4 \right\}, \\ \varepsilon_4 &= \min \left\{ \left(4C(\bar{\rho}) \right)^{-6}, 1 \right\}.\end{aligned}$$

Proof. Proposition 4.1 follows from Lemmas 4.2-4.8 below.

Lemma 4.2 *Let (ρ, u) be a smooth solution of (1.1)-(1.3) with $0 \leq \rho \leq 2\bar{\rho}$, $\bar{\rho} \geq \tilde{\rho} + 1$ and $0 < \gamma - 1 < 1$. Then there exists a positive constant C such that*

$$\int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^3 \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{4}} E_0. \quad (4.5)$$

Proof. A straightforward calculation implies that

$$G(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds = \frac{1}{\gamma - 1} [\rho^\gamma - \tilde{\rho}^\gamma - \gamma \tilde{\rho}^{\gamma-1} (\rho - \tilde{\rho})].$$

Next, we claim that

$$G(\rho) \geq \begin{cases} (\gamma - 1)^{-\frac{1}{4}} |\rho - \tilde{\rho}|^{\gamma-1}, & |\rho - \tilde{\rho}| > (\gamma - 1)^{\frac{1}{3}}, \\ (\gamma - 1)^{-\frac{1}{4}} |\rho - \tilde{\rho}|^3, & |\rho - \tilde{\rho}| \leq (\gamma - 1)^{\frac{1}{3}}. \end{cases} \quad (4.6)$$

Case 1: $|\rho - \tilde{\rho}| > (\gamma - 1)^{\frac{1}{3}}$.

Without loss of generality, we only consider the case of $\rho - \tilde{\rho} > (\gamma - 1)^{\frac{1}{3}}$. Now we define

$$f(\rho) = \rho^\gamma - \tilde{\rho}^\gamma - \gamma \tilde{\rho}^{\gamma-1} (\rho - \tilde{\rho}) - (\gamma - 1)^{\frac{3}{4}} (\rho - \tilde{\rho})^{\gamma-1},$$

and note that

$$f(\tilde{\rho}) = 0, \quad f'(\rho) = \gamma \rho^{\gamma-1} - \gamma \tilde{\rho}^{\gamma-1} - (\gamma - 1)^{\frac{7}{4}} (\rho - \tilde{\rho})^{\gamma-2}.$$

Thus

$$f(\rho) = f'(\xi)(\rho - \tilde{\rho}), \quad (4.7)$$

where $\xi = (1 - \theta)\rho + \theta\tilde{\rho}$ ($0 < \theta < 1$). Then

$$\begin{aligned}f'(\xi) &= \gamma \xi^{\gamma-1} - \gamma \tilde{\rho}^{\gamma-1} - (\gamma - 1)^{\frac{7}{4}} (\xi - \tilde{\rho})^{\gamma-2} \\ &= \gamma(\gamma - 1)(\xi - \tilde{\rho}) [(1 - \theta_1)\xi + \theta_1\tilde{\rho}]^{\gamma-2} - (\gamma - 1)^{\frac{7}{4}} (\xi - \tilde{\rho})^{\gamma-2} \quad (0 < \theta_1 < 1) \\ &= (\gamma - 1)(1 - \theta)(\rho - \tilde{\rho}) \left[\gamma((1 - \theta_1)\xi + \theta_1\tilde{\rho})^{\gamma-2} - (\gamma - 1)^{\frac{3}{4}} (\xi - \tilde{\rho})^{\gamma-3} \right] \\ &> (\gamma - 1)(1 - \theta)(\rho - \tilde{\rho}) \left[C\tilde{\rho}^{\gamma-2} - C(\gamma - 1)^{\frac{3}{4}} (\gamma - 1)^{\frac{\gamma-3}{3}} \right] \\ &> (\gamma - 1)(1 - \theta)(\rho - \tilde{\rho}) \left[C\tilde{\rho}^{\gamma-2} - C(\gamma - 1)^{\frac{1}{12}} \right] > 0, \quad (1 < \gamma < 2),\end{aligned}$$

when $\gamma - 1 \leq C\bar{\rho}^{12(\gamma-2)} \leq \frac{C}{\bar{\rho}^{12}} = \eta$. This together with (4.7) implies the first inequality of (4.6).

Case 2: $|\rho - \tilde{\rho}| \leq (\gamma - 1)^{\frac{1}{3}}$.

Similar to **Case 1**, we only consider the case of $0 \leq \rho - \tilde{\rho} \leq (\gamma - 1)^{\frac{1}{3}}$. Now let

$$g(\rho) = \rho^\gamma - \tilde{\rho}^\gamma - \gamma\tilde{\rho}^{\gamma-1}(\rho - \tilde{\rho}) - (\gamma - 1)^{\frac{3}{4}}(\rho - \tilde{\rho})^3,$$

and note that

$$g'(\rho) = \gamma\rho^{\gamma-1} - \gamma\tilde{\rho}^{\gamma-1} - 3(\gamma - 1)^{\frac{3}{4}}(\rho - \tilde{\rho})^2,$$

$$g''(\rho) = \gamma(\gamma - 1)\rho^{\gamma-2} - 6(\gamma - 1)^{\frac{3}{4}}(\rho - \tilde{\rho}).$$

Thus $g(\tilde{\rho}) = g'(\tilde{\rho}) = 0$, so that

$$g(\rho) = g''(\zeta)(\rho - \tilde{\rho})^2, \quad (4.8)$$

where $\zeta = (1 - \theta_2)\rho + \theta_2\tilde{\rho}$ ($0 < \theta_2 < 1$). Then

$$\begin{aligned} g''(\zeta) &= \gamma(\gamma - 1)\zeta^{\gamma-2} - 6(\gamma - 1)^{\frac{3}{4}}(\zeta - \tilde{\rho}) \\ &= \gamma(\gamma - 1)[(1 - \theta_2)\rho + \theta_2\tilde{\rho}]^{\gamma-2} - 6(1 - \theta_2)(\gamma - 1)^{\frac{3}{4}}(\rho - \tilde{\rho}) \\ &> (\gamma - 1)\left[C\bar{\rho}^{\gamma-2} - C(\gamma - 1)^{\frac{1}{12}}\right] \\ &> 0, \end{aligned}$$

when $\gamma - 1 \leq \eta$. This together with (4.8) implies the second inequality of (4.6). We thus obtain (4.6). Combining (3.6) and (4.6), we get

$$\begin{aligned} \int_{\Sigma_1} |\rho - \tilde{\rho}|^3 &\leq C(\bar{\rho}) \int_{\Sigma_1} |\rho - \tilde{\rho}|^{\gamma-1} \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{4}} \int_{\mathbb{R}^3} G(\rho) \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{4}} E_0, \\ \int_{\Sigma_2} |\rho - \tilde{\rho}|^3 &\leq (\gamma - 1)^{\frac{1}{4}} \int_{\mathbb{R}^3} G(\rho) \leq (\gamma - 1)^{\frac{1}{4}} E_0, \end{aligned}$$

where

$$\begin{cases} \Sigma_1 = \left\{x \in \mathbb{R}^3 : |\rho(x, t) - \tilde{\rho}| > (\gamma - 1)^{\frac{1}{3}}\right\}, \\ \Sigma_2 = \left\{x \in \mathbb{R}^3 : |\rho(x, t) - \tilde{\rho}| \leq (\gamma - 1)^{\frac{1}{3}}\right\}. \end{cases}$$

Thus

$$\int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^3 = \int_{\Sigma_1} |\rho - \tilde{\rho}|^3 + \int_{\Sigma_2} |\rho - \tilde{\rho}|^3 \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{4}} E_0.$$

On the other hand, for $1 > \gamma - 1 > \eta$, it is clear that $G(\rho) \geq C(\bar{\rho})(\rho - \tilde{\rho})^2$. Then

$$\int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^3 \leq C(\bar{\rho}) \int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^2 \leq C(\bar{\rho})\eta^{-\frac{1}{4}}(\gamma - 1)^{\frac{1}{4}} E_0 \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{4}} E_0.$$

This completes the prove of the Lemma 4.2.

Lemma 4.3 *Under the conditions of Proposition 4.1, we have*

$$\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{C\tilde{\rho}E_0}{\mu}, \quad (4.9)$$

$$\text{provided } \frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \frac{\tilde{\rho}}{2C}.$$

Proof. Multiplying (2.1)₂ by u and then integrating the resulting equality over \mathbb{R}^3 , and using integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2) = \int_{\mathbb{R}^3} (P - P(\tilde{\rho})) \operatorname{div} u. \quad (4.10)$$

Now, we turn to estimate the term on the right-hand side of (4.10),

$$\begin{aligned} \int_{\mathbb{R}^3} (P - P(\tilde{\rho})) \operatorname{div} u &\leq \|P - P(\tilde{\rho})\|_{L^3} \|\nabla u\|_{L^{\frac{3}{2}}} \\ &\leq \|P - P(\tilde{\rho})\|_{L^3} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{C}{\mu} \|P - P(\tilde{\rho})\|_{L^3}^{\frac{4}{3}} \|u\|_{L^2}^{\frac{2}{3}} + \frac{\mu}{16} \|\nabla u\|_{L^2}^2 \\ &\leq \frac{C}{\mu} (\gamma-1)^{\frac{1}{9}} E_0^{\frac{4}{9}} \|u\|_{L^2}^{\frac{2}{3}} + \frac{\mu}{16} \|\nabla u\|_{L^2}^2. \end{aligned} \quad (4.11)$$

Since

$$\begin{aligned} \tilde{\rho} \int_{\mathbb{R}^3} |u|^2 &\leq \int_{\mathbb{R}^3} |\rho - \tilde{\rho}| |u|^2 + \int_{\mathbb{R}^3} \rho |u|^2 \\ &\leq \left(\int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |u|^3 \right)^{\frac{2}{3}} + E_0 \\ &\leq C(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}} (\mu^{-\frac{1}{2}} \|u\|_{L^2}^2 + \mu^{\frac{1}{2}} \|\nabla u\|_{L^2}^2) + E_0 \\ &\leq \frac{C(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}}}{\mu^{\frac{1}{2}}} \|u\|_{L^2}^2 + C(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}} \mu^{\frac{1}{2}} \|\nabla u\|_{L^2}^2 + E_0 \\ &\leq \frac{\tilde{\rho}}{2} \|u\|_{L^2}^2 + C(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}} \mu^{\frac{1}{2}} \|\nabla u\|_{L^2}^2 + E_0, \end{aligned} \quad (4.12)$$

$$\text{provided } \frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \frac{\tilde{\rho}}{2C}, \quad \frac{(\gamma-1)^{\frac{1}{18}} E_0^{\frac{1}{12}}}{\mu^{\frac{1}{6}}} \leq 1. \text{ Then}$$

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^2 &\leq \frac{C(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}} \mu^{\frac{1}{2}} \|\nabla u\|_{L^2}^2}{\tilde{\rho}} + \frac{CE_0}{\tilde{\rho}} \\ &\leq C(\gamma-1)^{\frac{1}{18}} E_0^{\frac{1}{12}} \mu^{\frac{5}{6}} \|\nabla u\|_{L^2}^2 + \frac{C\mu^{\frac{1}{3}} E_0^{\frac{3}{4}}}{(\gamma-1)^{\frac{1}{36}}}. \end{aligned} \quad (4.13)$$

Combining (4.11) and (4.13), then using Young inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} (P - P(\tilde{\rho})) \operatorname{div} u &\leq \frac{C}{\mu} (\gamma - 1)^{\frac{1}{9}} E_0^{\frac{4}{9}} \left((\gamma - 1)^{\frac{1}{54}} E_0^{\frac{1}{36}} \mu^{\frac{5}{18}} \|\nabla u\|_{L^2}^{\frac{2}{3}} + \frac{\mu^{\frac{1}{9}} E_0^{\frac{1}{4}}}{(\gamma - 1)^{\frac{1}{108}}} \right) + \frac{\mu}{16} \|\nabla u\|_{L^2}^2 \\
&\leq \frac{C(\gamma - 1)^{\frac{7}{54}} E_0^{\frac{17}{36}}}{\mu^{\frac{13}{18}}} \|\nabla u\|_{L^2}^{\frac{2}{3}} + \frac{(\gamma - 1)^{\frac{11}{108}} E_0^{\frac{25}{36}}}{\mu^{\frac{8}{9}}} + \frac{\mu}{16} \|\nabla u\|_{L^2}^2 \\
&\leq \frac{C(\gamma - 1)^{\frac{7}{36}} E_0^{\frac{17}{24}}}{\mu^{\frac{19}{12}}} + \frac{C(\gamma - 1)^{\frac{11}{108}} E_0^{\frac{25}{36}}}{\mu^{\frac{8}{9}}} + \frac{\mu}{8} \|\nabla u\|_{L^2}^2. \tag{4.14}
\end{aligned}$$

Substituting (4.14) into (4.10) and integrating the resulting inequality over $[0, \sigma(T)]$, we get

$$\begin{aligned}
&\sup_{0 \leq t \leq \sigma(T)} \frac{1}{2} \int_{\mathbb{R}^3} \rho |u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 \right) \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} (\rho_0 - \tilde{\rho}) |u_0|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \tilde{\rho} |u_0|^2 + \frac{C(\gamma - 1)^{\frac{7}{36}} E_0^{\frac{17}{24}}}{\mu^{\frac{19}{12}}} + \frac{C(\gamma - 1)^{\frac{11}{108}} E_0^{\frac{25}{36}}}{\mu^{\frac{8}{9}}} \\
&\leq C \|\rho_0 - \tilde{\rho}\|_{L^3} \|u_0\|_{L^3}^2 + C \tilde{\rho} \|u_0\|^2 + \frac{C(\gamma - 1)^{\frac{7}{36}} E_0^{\frac{17}{24}}}{\mu^{\frac{19}{12}}} + \frac{C(\gamma - 1)^{\frac{11}{108}} E_0^{\frac{25}{36}}}{\mu^{\frac{8}{9}}} \\
&\leq C(\gamma - 1)^{\frac{1}{12}} E_0^{\frac{5}{6}} + C \tilde{\rho} E_0 + \frac{C(\gamma - 1)^{\frac{7}{36}} E_0^{\frac{17}{24}}}{\mu^{\frac{19}{12}}} + \frac{C(\gamma - 1)^{\frac{11}{108}} E_0^{\frac{25}{36}}}{\mu^{\frac{8}{9}}} \\
&\leq C(\gamma - 1)^{\frac{1}{13}} E_0^{\frac{25}{36}} \left((\gamma - 1)^{\frac{1}{156}} E_0^{\frac{5}{36}} + \frac{(\gamma - 1)^{\frac{7}{36} - \frac{1}{13}} E_0^{\frac{1}{72}}}{\mu^{\frac{19}{12}}} + \frac{(\gamma - 1)^{\frac{11}{108} - \frac{1}{13}}}{\mu^{\frac{8}{9}}} \right) + C \tilde{\rho} E_0. \tag{4.15}
\end{aligned}$$

Claim:

$$\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{C(\gamma - 1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{C \tilde{\rho} E_0}{\mu}, \tag{4.16}$$

where

$$E_{12} = 1 + \frac{1}{\mu} + \frac{(\gamma - 1)^{\frac{35}{1404}}}{\mu^{\frac{113}{117}}}. \tag{4.17}$$

In fact, (4.15) implies

$$\begin{aligned}
&\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \\
&\leq \frac{C(\gamma - 1)^{\frac{1}{13}} E_0^{\frac{25}{36}}}{\mu} \left((\gamma - 1)^{\frac{1}{156}} E_0^{\frac{5}{36}} + \frac{(\gamma - 1)^{\frac{7}{36} - \frac{1}{13}} E_0^{\frac{1}{72}}}{\mu^{\frac{19}{12}}} + \frac{(\gamma - 1)^{\frac{11}{108} - \frac{1}{13}}}{\mu^{\frac{8}{9}}} \right) + \frac{C \tilde{\rho} E_0}{\mu} \\
&= \frac{C(\gamma - 1)^{\frac{1}{13}} E_0^{\frac{25}{36}}}{\mu^{\frac{12}{13}}} \left(\frac{(\gamma - 1)^{\frac{1}{156}} E_0^{\frac{5}{36}}}{\mu^{\frac{1}{13}}} + \frac{(\gamma - 1)^{\frac{7}{36} - \frac{1}{13}} E_0^{\frac{1}{72}}}{\mu^{\frac{19}{12} + \frac{1}{13}}} + \frac{(\gamma - 1)^{\frac{11}{108} - \frac{1}{13}}}{\mu^{\frac{8}{9} + \frac{1}{13}}} \right) + \frac{C \tilde{\rho} E_0}{\mu}
\end{aligned}$$

$$\begin{aligned}
&= \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}}}{\mu^{\frac{12}{13}}} \left\{ \left(\frac{(\gamma-1) E_0^{\frac{5}{36} \times 156}}{\mu^{12}} \right)^{\frac{1}{156}} + \left(\frac{(\gamma-1) E_0^{\frac{1}{72} \times \frac{468}{55}}}{\mu^{\frac{103}{156} \times \frac{468}{55}}} \right)^{\frac{468}{55}} \frac{1}{\mu} \right. \\
&\quad \left. + \frac{(\gamma-1)^{\frac{11}{108} - \frac{1}{13}}}{\mu^{\frac{8}{9} + \frac{1}{13}}} \right\} + \frac{C\tilde{\rho}E_0}{\mu} \\
&\leq \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{C\tilde{\rho}E_0}{\mu}, \tag{4.18}
\end{aligned}$$

where (4.2) has been used. \square

Lemma 4.4 *Under the conditions of Proposition 4.1, we have*

$$\begin{aligned}
A_1(T) &\leq \sup_{0 \leq t \leq T} \left\{ \frac{4}{\mu} \int_{\mathbb{R}^3} \sigma \operatorname{div} u (P - P(\tilde{\rho})) \right\} + \frac{C(\gamma-1 + P(\tilde{\rho})) E_0}{\mu} \\
&\quad + \frac{C}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| |\nabla u|^2 + \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|_{L^3}^3 \tag{4.19}
\end{aligned}$$

and

$$\begin{aligned}
A_2(T) &\leq C A_1(\sigma(T)) + C \left(\frac{1}{\mu^2} + \frac{(\gamma-1)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 \\
&\quad + C \left(\frac{1}{\mu^2} + \frac{(\gamma-1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 + \frac{C P(\tilde{\rho})^2}{\mu^2} E_0. \tag{4.20}
\end{aligned}$$

Proof. The proof of Lemma 4.4 is similar to Lemma 3.4, we just need to deal with the first terms in (3.12) and (3.18) again. Here J_1 and J_2 denote I_1 and II_1 in Lemma 3.4 respectively. Integrating by parts gives

$$\begin{aligned}
J_1 &= - \int_{\mathbb{R}^3} \sigma^m \dot{u} \cdot \nabla P \\
&= \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u_t (P - P(\tilde{\rho})) + \int_{\mathbb{R}^3} \sigma^m \operatorname{div} (u \cdot \nabla u) (P - P(\tilde{\rho})) \\
&= \frac{d}{dt} \left(\int_{\mathbb{R}^3} \sigma^m \operatorname{div} u (P - P(\tilde{\rho})) \right) - m \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u (P - P(\tilde{\rho})) \\
&\quad + \int_{\mathbb{R}^3} \sigma^m \left((\gamma-1) P (\operatorname{div} u)^2 + (P - P(\tilde{\rho})) \partial_i u^j \partial_j u^i + P(\tilde{\rho}) (\operatorname{div} u)^2 \right) \tag{4.21}
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= - \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\partial_j P_t + \operatorname{div}(\partial_j P u)] \\
&= \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j P_t + \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j (\partial_j P u^k) \\
&= (1-\gamma) \int_0^T \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u \operatorname{div} u (P - P(\tilde{\rho})) - \gamma P(\tilde{\rho}) \int_0^T \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u \operatorname{div} u
\end{aligned}$$

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_j u^k (P - P(\tilde{\rho})) \\
& \leq \frac{\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |P - P(\tilde{\rho})|^4 \\
& \quad + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + \frac{CP(\tilde{\rho})^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^2. \tag{4.22}
\end{aligned}$$

Then, from (3.17) and (3.22), we have

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left(\frac{\mu}{4} \sigma^m \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \sigma^m \|\operatorname{div} u\|_{L^2}^2 \right) + \int_0^T \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 \\
& \leq \sup_{0 \leq t \leq T} \left\{ \int_{\mathbb{R}^3} \sigma^m \operatorname{div} u (P - P(\tilde{\rho})) \right\} - \int_0^{\sigma(T)} \int_{\mathbb{R}^3} m \sigma^{m-1} \operatorname{div} u (P - P(\tilde{\rho})) \\
& \quad + C \int_0^T \int_{\mathbb{R}^3} \sigma^m \left((\gamma-1) P (\operatorname{div} u)^2 + (P - P(\tilde{\rho})) |\nabla u|^2 + P(\tilde{\rho}) (\operatorname{div} u)^2 \right) \\
& \quad + C(2\mu + \lambda) \int_0^T \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 \tag{4.23}
\end{aligned}$$

and

$$\begin{aligned}
& \sigma^m \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + (\mu + \lambda) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\operatorname{div} \dot{u}|^2 \\
& \leq C \int_0^T \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |P - P(\tilde{\rho})|^4 \\
& \quad + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 \\
& \quad + \frac{CP(\tilde{\rho})^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^2. \tag{4.24}
\end{aligned}$$

Choosing $m = 1$ in (4.23) and $m = 3$ in (4.24), one gets (4.19) and (4.20). \square

Lemma 4.5 *Under the conditions of Proposition 4.1, it holds that*

$$\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} \leq E_{13}, \tag{4.25}$$

$$\sup_{0 \leq t \leq \sigma(T)} \sigma \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \leq E_{14}, \tag{4.26}$$

provided

$$\frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \min \left\{ \left(4C(\tilde{\rho}) \right)^{-6}, 1 \right\} \triangleq \varepsilon_4,$$

where

$$E_{13} = C(\bar{\rho}) + C(M + 1) + C\left(\frac{1}{\mu} + \gamma + P(\bar{\rho})\right)(E_{12} + 1), \quad (4.27)$$

$$\begin{aligned} E_{14} &= E_{13} + C(\bar{\rho})(1 + (\gamma - 1)^2) + C\left(\frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu}\right)^2 \frac{E_{13}^3}{\mu^4} \\ &\quad + C\left(\frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2}\right) \frac{E_{13}^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} + C\bar{\rho}^{2\gamma-1}. \end{aligned} \quad (4.28)$$

Proof. By an argument similar to the proof of Lemma 3.5, one can derive from (3.25) that

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \|\operatorname{div} u\|_{L^2}^2 - \int_{\mathbb{R}^3} \operatorname{div} u (P - P(\tilde{\rho})) \right) + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \\ &= \int_{\mathbb{R}^3} \rho \dot{u} (u \cdot \nabla u) - \int_{\mathbb{R}^3} \operatorname{div} u P_t \\ &\leq C(\bar{\rho}) \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \rho |u|^3 \right)^{\frac{1}{3}} \|\nabla u\|_{L^6} + \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div} ((P - P(\tilde{\rho}))u) \\ &\quad + (\gamma - 1) \int_{\mathbb{R}^3} P |\operatorname{div} u|^2 + P(\tilde{\rho}) \int_{\mathbb{R}^3} |\operatorname{div} u|^2 \\ &\leq C(\bar{\rho}) \left(\int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right) A_3^{\frac{1}{3}}(\sigma(T)) + C(\bar{\rho}) A_3^{\frac{1}{3}}(\sigma(T)) (\gamma - 1)^{\frac{1}{12}} E_0^{\frac{1}{3}} \\ &\quad + \frac{C}{2\mu + \lambda} \left(\|\nabla u\|_{L^2}^2 + \|P - P(\tilde{\rho})\|_{L^4}^4 \right) + \frac{C(\bar{\rho})}{(2\mu + \lambda)^2} \|P - P(\tilde{\rho})\|_{L^3}^2 \|\nabla u\|_{L^2}^2 \\ &\quad + \frac{1}{4} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C(\bar{\rho}) (\gamma - 1) \|\nabla u\|_{L^2}^2 + P(\tilde{\rho}) \|\nabla u\|_{L^2}^2. \end{aligned} \quad (4.29)$$

Integrating (4.29) over $(0, \sigma(T))$ and using (4.14) give that

$$\begin{aligned} &\frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \|\operatorname{div} u\|_{L^2}^2 - \int_{\mathbb{R}^3} \operatorname{div} u (P - P(\tilde{\rho})) + \frac{1}{2} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \\ &\leq C(\bar{\rho}) A_3^{\frac{1}{3}}(\sigma(T)) (\gamma - 1)^{\frac{1}{12}} E_0^{\frac{1}{3}} + \frac{C}{(2\mu + \lambda)} \int_0^{\sigma(T)} \|P - P(\tilde{\rho})\|_{L^4}^4 \\ &\quad + C \left(\frac{1}{(2\mu + \lambda)} + \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{2}{3}}}{(2\mu + \lambda)^2} + (\gamma - 1) + P(\tilde{\rho}) \right) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + C\mu(M + 1). \end{aligned}$$

provided $\frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}}\right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq (4C(\bar{\rho}))^{-6}.$

Then, using Lemma 3.2, Lemma 4.2 and (4.3), we have

$$\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu}$$

$$\begin{aligned}
&\leq \frac{C(\bar{\rho})(\gamma-1)^{\frac{1}{12}}E_0^{\frac{1}{3}}}{\mu} \left(\frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{6}} + \frac{C(\gamma-1)^{\frac{1}{4}}E_0}{\mu^2} \\
&\quad + \frac{C}{\mu} \left(\frac{1}{(2\mu+\lambda)} + \frac{(\gamma-1)^{\frac{1}{6}}E_0^{\frac{2}{3}}}{(2\mu+\lambda)^2} + (\gamma-1) + P(\tilde{\rho}) \right) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + C(M+1) \\
&\leq \frac{C(\bar{\rho})(\gamma-1)^{\frac{1}{12}}E_0^{\frac{1}{3}}}{\mu} + \frac{C(\gamma-1)^{\frac{1}{4}}E_0}{\mu^2} + C(M+1) \\
&\quad + \frac{C}{\mu} \left(\frac{1}{(2\mu+\lambda)} + \frac{(\gamma-1)^{\frac{1}{6}}E_0^{\frac{2}{3}}}{(2\mu+\lambda)^2} + (\gamma-1) + P(\tilde{\rho}) \right) \left(\frac{(\gamma-1)^{\frac{1}{13}}E_0^{\frac{25}{36}}E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right). \quad (4.30)
\end{aligned}$$

Using (4.2), we get

$$\begin{aligned}
&\frac{C(\bar{\rho})(\gamma-1)^{\frac{1}{12}}E_0^{\frac{1}{3}}}{\mu} + \frac{C(\gamma-1)^{\frac{1}{4}}E_0}{\mu^2} + C(M+1) \\
&\quad + \frac{C}{\mu} \left(\frac{1}{2\mu+\lambda} + \frac{(\gamma-1)^{\frac{1}{6}}E_0^{\frac{2}{3}}}{(2\mu+\lambda)^2} + (\gamma-1) + P(\tilde{\rho}) \right) \left(\frac{(\gamma-1)^{\frac{1}{13}}E_0^{\frac{25}{36}}E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right) \\
&= C(\bar{\rho}) \left(\frac{(\gamma-1)E_0^4}{\mu^{12}} \right)^{\frac{1}{12}} + C \left(\frac{(\gamma-1)E_0^4}{\mu^8} \right)^{\frac{1}{4}} + C(M+1) \\
&\quad + C \left(\frac{1}{2\mu+\lambda} + \left(\frac{(\gamma-1)E_0^4}{(2\mu+\lambda)^{12}} \right)^{\frac{1}{6}} + (\gamma-1) + P(\tilde{\rho}) \right) \left(\left(\frac{(\gamma-1)E_0^{\frac{325}{36}}}{\mu^{12}} \right)^{\frac{1}{13}} E_{12} + 1 \right) \\
&\leq C(\bar{\rho}) + C(M+1) + C \left(\frac{1}{\mu} + \gamma + P(\bar{\rho}) \right) (E_{12} + 1) = E_{13}. \quad (4.31)
\end{aligned}$$

Next taking $m = 1$ in (4.24), we have

$$\begin{aligned}
&\sigma \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 + (\mu + \lambda) \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\operatorname{div} \dot{u}|^2 \\
&\leq \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} \right) \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})|^4 \\
&\quad + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 + \frac{CP(\tilde{\rho})^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 \\
&\leq \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + C(\bar{\rho}) \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} \right) \|P - P(\tilde{\rho})\|_{L^3}^3 + \frac{CP(\tilde{\rho})^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 \\
&\quad + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 \\
&= \sum_{i=1}^4 K_i. \quad (4.32)
\end{aligned}$$

In fact, we just need to deal with K_3 . Using (2.12) and Cauchy inequality, we get

$$\begin{aligned}
K_3 &= C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 \\
&\leq \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \frac{C}{(2\mu+\lambda)^3} \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2} \int_0^{\sigma(T)} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^3 \\
&\quad + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \frac{C}{(2\mu+\lambda)^3} \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2} \int_0^{\sigma(T)} \sigma \|P - P(\tilde{\rho})\|_{L^6}^3 \\
&\leq \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \frac{CE_9^{\frac{1}{2}}}{(2\mu+\lambda)^3} \sup_{0 \leq t \leq \sigma(T)} \sigma^{\frac{1}{2}} \|\sqrt{\rho} \dot{u}\|_{L^2} \int_0^{\sigma(T)} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\
&\quad + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \frac{CE_9^{\frac{1}{2}}}{(2\mu+\lambda)^3} \int_0^{\sigma(T)} \sigma \|P - P(\tilde{\rho})\|_{L^6}^3 \\
&\leq \frac{1}{4} \sup_{0 \leq t \leq \sigma(T)} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right)^2 \frac{CE_{13}}{(2\mu+\lambda)^6} \left(\int_0^{\sigma(T)} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right)^2 \\
&\quad + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \frac{CE_9^{\frac{1}{2}}(\gamma-1)^{\frac{1}{8}}E_0^{\frac{1}{2}}}{(2\mu+\lambda)^3} \\
&\leq \frac{1}{4} \sup_{0 \leq t \leq \sigma(T)} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right)^2 \frac{CE_9^3\mu^2}{(2\mu+\lambda)^6} \\
&\quad + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right) \frac{CE_9^{\frac{1}{2}}(\gamma-1)^{\frac{1}{8}}E_0^{\frac{1}{2}}}{(2\mu+\lambda)^3}. \tag{4.33}
\end{aligned}$$

Substituting (4.33) into (4.32), and using (4.30), we have

$$\begin{aligned}
&\sup_{0 \leq t \leq \sigma(T)} \sigma \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \\
&\leq E_{13} + C(\bar{\rho}) \left(\frac{1}{\mu^2} + \frac{(\gamma-1)^2}{\mu^2} \right) (\gamma-1)^{\frac{1}{4}} E_0 + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right)^2 \frac{CE_9^3\mu}{(2\mu+\lambda)^6} \\
&\quad + \left(\frac{1}{\mu^2} + \frac{(\gamma-1)^2}{\mu^2} + \frac{(2\mu+\lambda)^2}{\mu^2} \right) \frac{CE_9^{\frac{1}{2}}(\gamma-1)^{\frac{1}{8}}E_0^{\frac{1}{2}}}{(2\mu+\lambda)^3} + \frac{CP(\tilde{\rho})^2 E_0}{\mu^3} \\
&\leq E_{13} + C(\bar{\rho}) (1 + (\gamma-1)^2) \left(\frac{(\gamma-1)E_0^4}{\mu^8} \right)^{\frac{1}{4}} + \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right)^2 \frac{CE_9^3}{\mu^5} \\
&\quad + \left(\frac{1}{\mu^2} + \frac{(\gamma-1)^2}{\mu^2} + \frac{(2\mu+\lambda)^2}{\mu^2} \right) \frac{CE_{13}^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \left(\frac{(\gamma-1)E_0^4}{\mu^{12}} \right)^{\frac{1}{8}} + C\tilde{\rho}^{2\gamma-1} \frac{\tilde{\rho}E_0}{\mu^3} \\
&\leq E_{13} + C(\bar{\rho}) (1 + (\gamma-1)^2) + C \left(\frac{1}{\mu} + \frac{(\gamma-1)^2}{\mu} + \frac{(2\mu+\lambda)^2}{\mu} \right)^2 \frac{E_9^3}{\mu^5}
\end{aligned}$$

$$+C \left(\frac{1}{\mu^2} + \frac{(\gamma-1)^2}{\mu^2} + \frac{(2\mu+\lambda)^2}{\mu^2} \right) \frac{E_{13}^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} + C\tilde{\rho}^{2\gamma-1} = E_{14}, \quad (4.34)$$

where (4.2) has been used. \square

Next, we will close the *a priori* assumption on $A_3(\sigma(T))$.

Lemma 4.6 *Under the conditions of Proposition 4.1, it holds that*

$$A_3(\sigma(T)) \leq \left\{ \frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right\}^{\frac{1}{2}}, \quad (4.35)$$

provided

$$\frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \min \left\{ (C(E_{15}E_{17} + E_{16}))^{-4}, \varepsilon_4 \right\} \triangleq \varepsilon_5. \quad (4.36)$$

Proof. Multiplying (2.1)₂ by $3|u|u$ and integrating the resulting equation over \mathbb{R}^3 , using Lemma 2.1, Lemma 2.2 and Hölder inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^3 \\ & \leq C\mu \int_{\mathbb{R}^3} |u| |\nabla u|^2 + C \int_{\mathbb{R}^3} |P - P(\tilde{\rho})| |u| |\nabla u| \\ & \leq C\mu \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^3} + C \|P - P(\tilde{\rho})\|_{L^3} \|u\|_{L^6} \|\nabla u\|_{L^2} \\ & \leq C\mu^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{5}{2}} \left(\|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{2}} + \|P - P(\tilde{\rho})\|_{L^6}^{\frac{1}{2}} \right) + C \|P - P(\tilde{\rho})\|_{L^3} \|\nabla u\|_{L^2}^2. \end{aligned} \quad (4.37)$$

Integrating (4.37) over $(0, \sigma(T))$ and using Hölder inequality, one gets

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \rho |u|^3 \\ & \leq C\mu^{\frac{1}{2}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{\frac{5}{2}} \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{2}} + C\mu^{\frac{1}{2}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{\frac{5}{2}} \|P - P(\tilde{\rho})\|_{L^6}^{\frac{1}{2}} \\ & \quad + C \int_0^{\sigma(T)} \|P - P(\tilde{\rho})\|_{L^3} \|\nabla u\|_{L^2}^2 + \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \rho_0 |u_0|^3 \\ & \leq C\mu^{\frac{1}{2}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^4 \right)^{\frac{1}{2}} \left(\int_0^{\sigma(T)} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right)^{\frac{1}{4}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{4}} \\ & \quad + C\mu^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} \|P - P(\tilde{\rho})\|_{L^6}^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^{\frac{1}{2}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \\ & \quad + C \sup_{0 \leq t \leq \sigma(T)} \|P - P(\tilde{\rho})\|_{L^3} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^3} |\rho_0 - \tilde{\rho}| |u_0|^3 + \int_{\mathbb{R}^3} \tilde{\rho} |u_0|^3 \end{aligned}$$

$$\begin{aligned}
&\leq C\mu^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{3}{4}} \left(\int_0^{\sigma(T)} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right)^{\frac{1}{4}} \\
&\quad + C\mu^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} \|P - P(\tilde{\rho})\|_{L^6}^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^{\frac{1}{2}} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \\
&\quad + C \sup_{0 \leq t \leq \sigma(T)} \|P - P(\tilde{\rho})\|_{L^3} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + C \|\rho_0 - \tilde{\rho}\|_{L^3} \|u_0\|_{L^6} \|u_0\|_{L^4}^2 \\
&\quad + C\tilde{\rho} \|u_0\|_{L^3}^3.
\end{aligned} \tag{4.38}$$

By Lemma 3.2, Lemma 4.2, Lemma 4.3 and Lemma 4.5, we obtain

$$\begin{aligned}
&\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \frac{\rho|u|^3}{\mu^3} \\
&\leq \frac{C}{\mu^2} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{3}{4}} \left(\frac{E_{13}^{\frac{3}{4}}}{\mu^{\frac{1}{4}}} + \frac{E_{13}^{\frac{1}{4}}(\gamma-1)^{\frac{1}{48}} E_0^{\frac{1}{12}}}{\mu^{\frac{1}{2}}} \frac{E_0^{\frac{1}{4}}}{\mu^{\frac{1}{4}}} + \frac{(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}}}{\mu} \frac{E_0^{\frac{1}{4}}}{\mu^{\frac{1}{4}}} \right) \\
&\quad + \frac{C}{\mu^3} \|\rho_0 - \tilde{\rho}\|_{L^3} \|\nabla u_0\|_{L^2}^{\frac{5}{2}} \|u_0\|_{L^2}^{\frac{1}{2}} + \frac{C\tilde{\rho}}{\mu^3} \|u_0\|_{L^2}^{\frac{3}{2}} \|\nabla u_0\|_{L^2}^{\frac{3}{2}} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho} E_0}{\mu} \right)^{\frac{3}{4}} E_{15} + \frac{C(\gamma-1)^{\frac{1}{12}} E_0^{\frac{7}{12}} M^{\frac{5}{4}}}{\mu^3} + \frac{C\tilde{\rho} E_0^{\frac{3}{4}} M^{\frac{3}{4}}}{\mu^3} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho} E_0}{\mu} \right)^{\frac{3}{4}} E_{15} + C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{4}} E_{16} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{4}} (E_{15} E_{17} + E_{16}) \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}},
\end{aligned} \tag{4.39}$$

provided $\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq (C(E_{15} E_{17} + E_{16}))^{-4}$, where

$$E_{15} = \frac{E_{13}^{\frac{3}{4}}}{\mu^{\frac{9}{4}}} + \frac{E_{13}^{\frac{1}{4}}}{\mu^{\frac{5}{2}}} + \frac{1}{\mu^{\frac{9}{4}}}, \quad E_{16} = \frac{M^{\frac{5}{4}}}{\mu^2} + M^{\frac{3}{4}}, \quad E_{17} = \left(E_{12} + \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu} \right)^{\frac{3}{4}}. \tag{4.40}$$

In what follows, we give the details of the calculations of E_{15} - E_{17} . (4.2) gives

$$E_{15} = \frac{1}{\mu^2} \left(\frac{E_{13}^{\frac{3}{4}}}{\mu^{\frac{1}{4}}} + \frac{E_{13}^{\frac{1}{4}}(\gamma-1)^{\frac{1}{48}} E_0^{\frac{1}{12}}}{\mu^{\frac{1}{2}}} \frac{E_0^{\frac{1}{4}}}{\mu^{\frac{1}{4}}} + \frac{(\gamma-1)^{\frac{1}{12}} E_0^{\frac{1}{3}}}{\mu} \frac{E_0^{\frac{1}{4}}}{\mu^{\frac{1}{4}}} \right)$$

$$\begin{aligned}
&= \frac{1}{\mu^2} \left(\frac{E_{13}^{\frac{3}{4}}}{\mu^{\frac{1}{4}}} + \frac{E_{13}^{\frac{1}{4}}}{\mu^{\frac{1}{2}}} \left(\frac{(\gamma-1)E_0^{16}}{\mu^{12}} \right)^{\frac{1}{48}} + \frac{1}{\mu^{\frac{1}{4}}} \left(\frac{(\gamma-1)E_0^7}{\mu^{12}} \right)^{\frac{1}{12}} \right) \\
&= \frac{E_{13}^{\frac{3}{4}}}{\mu^{\frac{9}{4}}} + \frac{E_{13}^{\frac{1}{4}}}{\mu^{\frac{5}{2}}} + \frac{1}{\mu^{\frac{9}{4}}}, \tag{4.41}
\end{aligned}$$

$$\begin{aligned}
E_{16} &= \frac{(\gamma-1)E_0^{\frac{19}{48}}M^{\frac{5}{4}}}{\mu^{\frac{11}{4}}} + \frac{\tilde{\rho}^{\frac{7}{8}}E_0^{\frac{9}{16}}M^{\frac{3}{4}}}{\mu^{\frac{11}{4}}} \\
&= \left(\frac{(\gamma-1)E_0^{\frac{19}{3}}}{\mu^{12}} \right)^{\frac{1}{16}} \frac{M^{\frac{5}{4}}}{\mu^2} + M^{\frac{3}{4}} \left(\frac{\tilde{\rho}E_0^{\frac{9}{14}}}{\mu^{\frac{22}{7}}} \right)^{\frac{7}{8}} \\
&= \frac{M^{\frac{5}{4}}}{\mu^2} + M^{\frac{3}{4}} \tag{4.42}
\end{aligned}$$

and

$$\begin{aligned}
&C \left(\frac{(\gamma-1)^{\frac{1}{13}}E_0^{\frac{25}{36}}E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right)^{\frac{3}{4}} \\
&= C \left(\frac{(\gamma-1)^{\frac{1}{36}}E_0^{\frac{1}{4}}(\gamma-1)^{\frac{23}{468}}E_0^{\frac{4}{9}}E_{12}}{\mu^{\frac{1}{3}}\mu^{\frac{23}{39}}} + \left(\frac{\tilde{\rho}^{\frac{1}{6}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^4 \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu} \right)^{\frac{3}{4}} \\
&= C \left(\frac{(\gamma-1)^{\frac{1}{36}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \left(\frac{(\gamma-1)E_0^{\frac{4}{9} \times \frac{468}{23}}}{\mu^{12}} \right)^{\frac{23}{468}} E_{12} + \left(\frac{\tilde{\rho}^{\frac{1}{6}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^4 \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu} \right)^{\frac{3}{4}} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{4}} \left(E_{12} + \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu} \right)^{\frac{3}{4}} \\
&= C \left(\frac{(\gamma-1)^{\frac{1}{36}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}}E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{4}} E_{17}. \tag{4.43}
\end{aligned}$$

□

Lemma 4.7 *Under the conditions of Proposition 4.1, it holds that*

$$A_1(T) \leq \left(\frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{8}}, \tag{4.44}$$

$$A_2(T) \leq \frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}}, \tag{4.45}$$

provided

$$\frac{\left((\gamma - 1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}}\right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \varepsilon_6,$$

where

$$\varepsilon_6 = \min \left\{ \left(C(E_{18} + E_{19} + E_{20}) \right)^{-17}, \left(C(E_{18} + E_{19} + E_{21}) \right)^{-8}, \varepsilon_5 \right\}$$

and E_{18} - E_{21} are given by (4.62), (4.64), (4.66) and (4.74) respectively.

Proof. First, we will prove (4.45). Recalling (4.20), we have

$$\begin{aligned} A_2(T) &\leq C A_1(\sigma(T)) + C \left(\frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 \\ &\quad + C \left(\frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 + \frac{C P(\tilde{\rho})^2}{\mu^2} E_0. \end{aligned} \quad (4.46)$$

Now, we turn to estimate the term $\int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4$. Due to (2.11),

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 \\ &\leq C \left(\frac{1}{(2\mu + \lambda)^2} + \frac{1}{\mu^2} \right) \int_0^T \sigma^3 \|\nabla u\|_{L^3}^2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \|P - P(\tilde{\rho})\|_{L^3}^2 \\ &\quad + \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|P - P(\tilde{\rho})\|_{L^4}^4 = \sum_{i=1}^3 L_i. \end{aligned} \quad (4.47)$$

By using Hölder inequality, Young inequality and (4.3), L_1 can be estimated as follows,

$$\begin{aligned} L_1 &= C \left(\frac{1}{(2\mu + \lambda)^2} + \frac{1}{\mu^2} \right) \int_0^T \sigma^3 \|\nabla u\|_{L^3}^2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\ &\leq C \left(\frac{1}{(2\mu + \lambda)^2} + \frac{1}{\mu^2} \right) \int_0^T \sigma^3 \|\nabla u\|_{L^2}^{\frac{2}{3}} \|\nabla u\|_{L^4}^{\frac{4}{3}} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\ &\leq \frac{C}{\mu^2} \left(\int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 \right)^{\frac{1}{3}} \left(\int_0^T \sigma^3 \|\nabla u\|_{L^2} \|\sqrt{\rho} \dot{u}\|_{L^2}^3 \right)^{\frac{2}{3}} \\ &\leq \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 + \frac{C}{\mu^3} \int_0^T \sigma^3 \|\nabla u\|_{L^2} \|\sqrt{\rho} \dot{u}\|_{L^2}^3 \\ &\leq \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 + \frac{C}{\mu^3} \sup_{0 \leq t \leq T} \left(\sigma^{\frac{1}{2}} \|\nabla u\|_{L^2} \right) \sup_{0 \leq t \leq T} \left(\sigma^{\frac{3}{2}} \|\sqrt{\rho} \dot{u}\|_{L^2} \right) \int_0^T \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\ &\leq \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 + \frac{C}{\mu^3} A_1^{\frac{3}{2}}(T) A_2^{\frac{1}{2}}(T). \end{aligned} \quad (4.48)$$

It follows from Lemma 4.2 and (4.3) that

$$\begin{aligned}
L_2 &= \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \|P - P(\tilde{\rho})\|_{L^3}^2 \\
&\leq \frac{C}{(2\mu + \lambda)^4} \sup_{0 \leq t \leq T} (\|P - P(\tilde{\rho})\|_{L^3}^2) \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\
&\leq \frac{C(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{2}{3}} A_1(T)}{(2\mu + \lambda)^4}.
\end{aligned} \tag{4.49}$$

Following a process similar to [16], we focus on estimating the term $\int_0^T \|P - P(\tilde{\rho})\|_{L^4}^4$. One deduces from (2.1)₁ that $P - P(\tilde{\rho})$ satisfies

$$(P - P(\tilde{\rho}))_t + u \cdot \nabla(P - P(\tilde{\rho})) + \gamma(P - P(\tilde{\rho})) \operatorname{div} u + \gamma P(\tilde{\rho}) \operatorname{div} u = 0. \tag{4.50}$$

Multiplying (4.50) by $3\sigma^3(P - P(\tilde{\rho}))^2$ and integrating the resulting equality over $\mathbb{R}^3 \times [0, T]$, using $\operatorname{div} u = \frac{1}{2\mu + \lambda} (G + P - P(\tilde{\rho}))$, we get

$$\begin{aligned}
&\frac{3\gamma - 1}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 \\
&= \int_{\mathbb{R}^3} \sigma^3 (P - P(\tilde{\rho}))^3 - \frac{(3\gamma - 1)}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 (P - P(\tilde{\rho}))^3 G + 3 \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma^2 (P - P(\tilde{\rho}))^3 \\
&\quad - 3\gamma P(\tilde{\rho}) \int_0^T \int_{\mathbb{R}^3} \sigma^3 (P - P(\tilde{\rho}))^2 \operatorname{div} u \\
&\leq C \sup_{0 \leq t \leq T} (\|P - P(\tilde{\rho})\|_{L^3}^3) + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma^2 (P - P(\tilde{\rho}))^3 + \frac{C(3\gamma - 1)}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |G|^4 \\
&\quad + \frac{3\gamma - 1}{2(2\mu + \lambda)} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 + \frac{CP(\tilde{\rho})^2(2\mu + \lambda)}{3\gamma - 1} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^2 \\
&\leq C(\gamma - 1)^{\frac{1}{4}} E_0 + \frac{C}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |G|^4 + \frac{3\gamma - 1}{2(2\mu + \lambda)} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 \\
&\quad + \frac{C(2\mu + \lambda)P^2(\tilde{\rho})E_0}{\mu}.
\end{aligned} \tag{4.51}$$

It follows from (2.9) that

$$\begin{aligned}
L_3 &= \frac{C(\gamma - 1)^{\frac{1}{4}} E_0}{(2\mu + \lambda)^3} + \frac{C}{(2\mu + \lambda)^2} \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \|\nabla u\|_{L^3}^2 \\
&\quad + \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \|P - P(\tilde{\rho})\|_{L^3}^2 + \frac{CP(\tilde{\rho})^2 E_0}{(2\mu + \lambda)^2 \mu} \\
&\leq \frac{C(\gamma - 1)^{\frac{1}{4}} E_0}{(2\mu + \lambda)^3} + \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 + \frac{1}{(2\mu + \lambda)^3} \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(2\mu + \lambda)^4} \sup_{0 \leq t \leq T} (\|P - P(\tilde{\rho})\|_{L^3}^2) \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \frac{CP(\tilde{\rho})^2 E_0}{(2\mu + \lambda)^2 \mu} \\
& \leq \frac{C(\gamma - 1)^{\frac{1}{4}} E_0}{(2\mu + \lambda)^3} + \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 + \frac{1}{(2\mu + \lambda)^4} \sup_{0 \leq t \leq T} (\|P - P(\tilde{\rho})\|_{L^3}^2) \int_0^T \sigma^3 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\
& \quad + \frac{1}{(2\mu + \lambda)^3} \sup_{0 \leq t \leq T} \left(\sigma^{\frac{1}{2}} \|\nabla u\|_{L^2} \right) \sup_{0 \leq t \leq T} \left(\sigma^{\frac{3}{2}} \|\sqrt{\rho} \dot{u}\|_{L^2} \right) \int_0^T \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \frac{CP(\tilde{\rho})^2 E_0}{(2\mu + \lambda)^2 \mu} \\
& \leq \frac{C(\gamma - 1)^{\frac{1}{4}} E_0}{(2\mu + \lambda)^3} + \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 + \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{2}{3}} A_1(T)}{(2\mu + \lambda)^4} + \frac{A_1^{\frac{3}{2}}(T) A_2^{\frac{1}{2}}(T)}{(2\mu + \lambda)^3} \\
& \quad + \frac{CP(\tilde{\rho})^2 E_0}{(2\mu + \lambda)^2 \mu}. \tag{4.52}
\end{aligned}$$

Substituting (4.48)-(4.52) into (4.47) shows that

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 \\
& \leq \frac{C(\gamma - 1)^{\frac{1}{4}} E_0}{\mu^3} + \frac{CA_1^{\frac{3}{2}}(T) A_2^{\frac{1}{2}}(T)}{\mu^3} + \frac{C(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{2}{3}} A_1(T)}{\mu^4} + \frac{CP(\tilde{\rho})^2 E_0}{\mu^3}. \tag{4.53}
\end{aligned}$$

And also we get

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 & \leq C\mu(\gamma - 1)^{\frac{1}{4}} E_0 + C\mu A_1^{\frac{3}{2}}(T) A_2^{\frac{1}{2}}(T) \\
& \quad + C(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{2}{3}} A_1(T) + C \frac{(2\mu + \lambda)^2}{\mu} P^2(\tilde{\rho}) E_0. \tag{4.54}
\end{aligned}$$

Next, we turn to estimate $A_1(\sigma(T))$. (4.19) shows that

$$\begin{aligned}
A_1(\sigma(T)) & \leq \frac{4}{\mu} \int_{\mathbb{R}^3} \sigma \operatorname{div} u (P - P(\tilde{\rho})) + \frac{C(\gamma - 1) E_0}{\mu} + \frac{CP(\tilde{\rho}) E_0}{\mu} \\
& \quad + \frac{C(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^3 + \frac{C}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| \|\nabla u\|^2. \tag{4.55}
\end{aligned}$$

Based on Lemma 2.2, Lemma 4.5 and (4.3), the last two terms in the right hand side of (4.55) can be estimated as follows:

$$\begin{aligned}
& \frac{C(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^3 \\
& \leq C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{L^6}^{\frac{3}{2}} \\
& \leq \frac{C}{\mu^{\frac{3}{2}}} \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^{\frac{3}{2}} (\|\sqrt{\rho} \dot{u}\|_{L^2} + \|P - P(\tilde{\rho})\|_{L^6})^{\frac{3}{2}} \\
& \leq \frac{C}{\mu^{\frac{3}{2}}} \sup_{0 \leq t \leq \sigma(T)} \left(\sigma^{\frac{1}{4}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \right) \sup_{0 \leq t \leq \sigma(T)} \left(\|\nabla u\|_{L^2}^{\frac{1}{2}} \right) \left(\int_0^{\sigma(T)} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right)^{\frac{3}{4}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{4}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{C}{\mu^{\frac{3}{2}}} \sup_{0 \leq t \leq \sigma(T)} \left(\|P - P(\tilde{\rho})\|_{L^6} \right)^{\frac{3}{2}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{3}{4}} \\
& \leq \frac{CA_1(T)E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{4}} + \frac{C(\gamma-1)^{\frac{1}{16}}E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{3}{4}}
\end{aligned} \tag{4.56}$$

and

$$\frac{C}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| |\nabla u|^2 \leq \frac{C(\tilde{\rho})}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2. \tag{4.57}$$

Substituting (4.56)-(4.57) into (4.55), one has

$$\begin{aligned}
A_1(\sigma(T)) & \leq \sup_{0 \leq t \leq \sigma(T)} \left\{ \frac{4}{\mu} \int_{\mathbb{R}^3} \sigma \operatorname{div} u (P - P(\tilde{\rho})) \right\} + \frac{C(\gamma-1)E_0}{\mu} + \frac{CP(\tilde{\rho})E_0}{\mu} \\
& + \frac{CA_1(T)E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{4}} + \frac{C(\gamma-1)^{\frac{1}{16}}E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{3}{4}} \\
& + \frac{C(\tilde{\rho})}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2.
\end{aligned} \tag{4.58}$$

It follows from (4.14) that

$$\begin{aligned}
A_1(\sigma(T)) & \leq \frac{C(\gamma-1)^{\frac{7}{36}}E_0^{\frac{17}{24}}}{\mu^{\frac{31}{12}}} + \frac{C(\gamma-1)^{\frac{11}{108}}E_0^{\frac{25}{36}}}{\mu^{\frac{17}{9}}} + \frac{C(\gamma-1)E_0}{\mu} + \frac{CP(\tilde{\rho})E_0}{\mu} \\
& + \frac{CA_1(T)E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{4}} + \frac{C(\gamma-1)^{\frac{1}{16}}E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \right)^{\frac{3}{4}} \\
& + \frac{C(\tilde{\rho})}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2.
\end{aligned} \tag{4.59}$$

Collecting (4.9), (4.46), (4.52), (4.53) and (4.59) implies that

$$\begin{aligned}
A_2(T) & \leq \underbrace{\frac{C(\gamma-1)^{\frac{7}{36}}E_0^{\frac{17}{24}}}{\mu^{\frac{31}{12}}} + \frac{C(\gamma-1)^{\frac{11}{108}}E_0^{\frac{25}{36}}}{\mu^{\frac{17}{9}}} + \frac{C(\gamma-1)E_0}{\mu} + \frac{CP(\tilde{\rho})E_0}{\mu}}_{N_1} \\
& + \underbrace{\frac{CA_1(T)E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\frac{(\gamma-1)^{\frac{1}{13}}E_0^{\frac{25}{36}}E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right)^{\frac{1}{4}}}_{N_2} \\
& + \underbrace{\frac{C(\gamma-1)^{\frac{1}{16}}E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\frac{(\gamma-1)^{\frac{1}{13}}E_0^{\frac{25}{36}}E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right)^{\frac{3}{4}}}_{N_3}
\end{aligned}$$

$$\begin{aligned}
& + \underbrace{\frac{C(\bar{\rho})(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{25}{13}}} + \frac{C(\bar{\rho})\tilde{\rho}E_0}{\mu^2}}_{N_4} + \underbrace{\left(1 + \frac{1}{\mu^4} + \frac{(\gamma-1)^2}{\mu^4} + \frac{(2\mu+\lambda)^2}{\mu^4}\right)}_{N_5} \\
& \times \underbrace{\left(\frac{C(\gamma-1)^{\frac{1}{4}} E_0}{\mu} \frac{C A_1^{\frac{3}{2}}(T) A_2^{\frac{1}{2}}(T)}{\mu} + \frac{C(\gamma-1)^{\frac{1}{6}} E_0^{\frac{2}{3}} A_1(T)}{\mu^2} + \frac{CP(\bar{\rho})^2 E_0}{\mu}\right)}_{N_6} \\
& + \underbrace{\frac{CP(\bar{\rho})E_0}{\mu^2}}_{N_7}. \tag{4.60}
\end{aligned}$$

Next we focus on dealing with N_1 - N_6 . In fact, (4.2) leads to

$$\begin{aligned}
N_1 + N_2 & \leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left\{ \frac{(\gamma-1)^{\frac{101}{612}} E_0^{\frac{181}{408}}}{\mu^{\frac{455}{204}}} + \frac{(\gamma-1)^{\frac{133}{1836}} E_0^{\frac{263}{612}}}{\mu^{\frac{235}{153}}} \right. \\
& + \frac{(\gamma-1)^{\frac{33}{34}} E_0^{\frac{25}{34}}}{\mu^{\frac{11}{17}}} + \left. \frac{(\gamma-1)^{\frac{5}{21216}} E_0^{\frac{13}{4896}} E_{12}^{\frac{1}{4}} E_{13}^{\frac{1}{4}}}{\mu^{\frac{2657}{1768}}} \right\} + C \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^2 \left(\frac{\tilde{\rho} E_0^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} \right)^{\frac{2}{9}} \tilde{\rho}^{\gamma-\frac{5}{9}} \\
& + C \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{9}{8}} \frac{\tilde{\rho}^{\frac{1}{12}}}{\mu^{\frac{17}{12}}} \\
& = C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left\{ \left(\frac{(\gamma-1) E_0^{\frac{543}{202}}}{\mu^{12}} \right)^{\frac{101}{612}} \frac{1}{\mu^{\frac{1}{4}}} + \left(\frac{(\gamma-1) E_0^{\frac{789}{133}}}{\mu^{12}} \right)^{\frac{133}{1836}} \frac{1}{\mu^{\frac{2}{3}}} \right. \\
& + \left(\frac{(\gamma-1) E_0^{\frac{25}{33}}}{\mu^{\frac{2}{3}}} \right)^{\frac{33}{34}} + \left(\frac{(\gamma-1) E_0^{\frac{169}{15}}}{\mu^{12}} \right)^{\frac{5}{21216}} \frac{E_{12}^{\frac{1}{4}} E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} + \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{16}{17}} \left(\frac{\tilde{\rho} E_0^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} \right)^{\frac{2}{9}} \tilde{\rho}^{\gamma-\frac{5}{9}} \\
& + \left. \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{136}} \frac{\tilde{\rho}^{\frac{1}{12}}}{\mu^{\frac{17}{12}}} \right\} \\
& \leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{18}, \tag{4.61}
\end{aligned}$$

where

$$E_{18} = \frac{1}{\mu^{\frac{1}{4}}} + \frac{1}{\mu^{\frac{2}{3}}} + 1 + \frac{E_{12}^{\frac{1}{4}} E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} + \tilde{\rho}^{\gamma-\frac{5}{9}} + \frac{\tilde{\rho}^{\frac{1}{12}}}{\mu^{\frac{17}{12}}}. \tag{4.62}$$

$$N_3 + N_4 \leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left\{ \frac{(\gamma-1)^{\frac{321}{3536}} E_0^{\frac{413}{816}}}{\mu^{\frac{813}{442}}} + \frac{(\gamma-1)^{\frac{21}{442}} E_0^{\frac{263}{612}}}{\mu^{\frac{347}{221}}} \right\}$$

$$\begin{aligned}
& + \left(\frac{(\gamma-1)E_0^4}{\mu^{12}} \right)^{\frac{1}{16}} \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{33}{17}} \frac{\tilde{\rho}^{\frac{1}{4}}}{\mu^{\frac{1}{2}}} + \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{50}{17}} \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu^{\frac{2}{3}}} \Big\} \\
& = C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left\{ \left(\frac{(\gamma-1)E_0^{\frac{5369}{963}}}{\mu^{12}} \right)^{\frac{321}{3536}} \frac{1}{\mu^{\frac{3}{4}}} \right. \\
& \quad \left. + \left(\frac{(\gamma-1)E_0^{\frac{263}{612} \times \frac{442}{21}}}{\mu^{12}} \right)^{\frac{21}{442}} \frac{1}{\mu} + \frac{\tilde{\rho}^{\frac{1}{4}}}{\mu^{\frac{1}{2}}} + \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu^{\frac{2}{3}}} \right\} \\
& \leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{19}, \tag{4.63}
\end{aligned}$$

where

$$E_{19} = \frac{1}{\mu^{\frac{3}{4}}} + \frac{1}{\mu} + \frac{\tilde{\rho}^{\frac{1}{4}}}{\mu^{\frac{1}{2}}} + \frac{\tilde{\rho}^{\frac{1}{3}}}{\mu^{\frac{2}{3}}}. \tag{4.64}$$

$$\begin{aligned}
N_5 \times N_6 + N_7 & \leq C \left(1 + \frac{1}{\mu^4} + \frac{(\gamma-1)^2}{\mu^4} + \frac{(2\mu+\lambda)^2}{\mu^4} \right) \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \\
& \quad \times \left\{ \frac{(\gamma-1)^{\frac{15}{68}} E_0^{\frac{25}{34}}}{\mu^{\frac{11}{17}}} + \frac{(\gamma-1)^{\frac{1}{9792}} E_0^{\frac{1088}{1088}}}{\mu^{\frac{817}{816}}} + \frac{(\gamma-1)^{\frac{7}{51}} E_0^{\frac{41}{102}}}{\mu^{\frac{28}{17}}} \right. \\
& \quad \left. + \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{16}{17}} \left(\frac{\tilde{\rho} E_0^{\frac{3}{2}}}{\mu} \right)^{\frac{1}{3}} \tilde{\rho}^{2\gamma-\frac{2}{3}} \right\} + C \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^4 \frac{\tilde{\rho}^{\gamma-\frac{2}{3}}}{\mu^{\frac{2}{3}}} \\
& = C \left(1 + \frac{1}{\mu^4} + \frac{(\gamma-1)^2}{\mu^4} + \frac{(2\mu+\lambda)^2}{\mu^4} \right) \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \\
& \quad \times \left\{ \left(\frac{(\gamma-1)E_0^{\frac{10}{3}}}{\mu^{\frac{44}{15}}} \right)^{\frac{15}{68}} + \left(\frac{(\gamma-1)E_0^9}{\mu^{12}} \right)^{\frac{1}{36} \times \frac{1}{272}} \frac{1}{\mu} + \left(\frac{(\gamma-1)E_0^{\frac{41}{14}}}{\mu^{12}} \right)^{\frac{7}{51}} \right\} \\
& \quad + C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{50}{17}} \frac{\tilde{\rho}^{\gamma-\frac{2}{3}}}{\mu^{\frac{2}{3}}} \\
& \leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{20}, \tag{4.65}
\end{aligned}$$

$$E_{20} = \left(1 + \frac{1}{\mu^4} + \frac{(\gamma-1)^2}{\mu^4} + \frac{(2\mu+\lambda)^2}{\mu^4} \right) \left(1 + \frac{1}{\mu} \right) + \frac{\tilde{\rho}^{\gamma-\frac{2}{3}}}{\mu^{\frac{2}{3}}}. \tag{4.66}$$

It thus follows (4.61), (4.63) and (4.65) that

$$\begin{aligned} A_2(T) &\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} (E_{18} + E_{19} + E_{20}) \\ &\leq \frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}}, \end{aligned} \quad (4.67)$$

provided $\left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right) \leq \left(C(E_{18} + E_{19} + E_{20}) \right)^{-17}.$

Finally, to finish the proof of Lemma 4.7, it remains to prove (4.44). With (4.19) and (4.58) at hand, we just have to estimate the terms $\frac{2\mu + \lambda}{\mu} \int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^3}^3$ and $\frac{C}{\mu} \int_{\sigma(T)}^T \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| |\nabla u|^2$. By Hölder inequality, we have

$$\begin{aligned} \frac{2\mu + \lambda}{\mu} \int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^3}^3 &\leq C \left(\int_{\sigma(T)}^T \|\nabla u\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\int_{\sigma(T)}^T \|\nabla u\|_{L^4}^4 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{CE_0}{\mu} \right)^{\frac{1}{2}} \left(\int_{\sigma(T)}^T \|\nabla u\|_{L^4}^4 \right)^{\frac{1}{2}} \end{aligned} \quad (4.68)$$

and

$$\begin{aligned} &\frac{C}{\mu} \int_{\sigma(T)}^T \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| |\nabla u|^2 \\ &\leq \frac{C}{\mu} \left(\int_{\sigma(T)}^T \int_{\mathbb{R}^3} |P - P(\tilde{\rho})|^4 \right)^{\frac{1}{4}} \left(\int_{\sigma(T)}^T \int_{\mathbb{R}^3} |\nabla u|^4 \right)^{\frac{1}{4}} \left(\int_{\sigma(T)}^T \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.69)$$

It follows from Lemma 4.3, (4.51), (4.53), Lemma 4.4, (4.68) and (4.69) that

$$\begin{aligned} A_1(T) &\leq A_1(\sigma(T)) + \frac{C(2\mu + \lambda)}{\mu} \int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^3}^3 + \frac{C}{\mu} \int_{\sigma(T)}^T \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| |\nabla u|^2 \\ &\leq \underbrace{\frac{C(\gamma-1)^{\frac{7}{36}} E_0^{\frac{17}{24}}}{\mu^{\frac{31}{12}}} + \frac{C(\gamma-1)^{\frac{11}{108}} E_0^{\frac{25}{36}}}{\mu^{\frac{17}{9}}} + \frac{C(\gamma-1)E_0}{\mu} + \frac{CP(\tilde{\rho})E_0}{\mu}}_{N_8} \\ &\quad + \underbrace{\frac{CA_1(T)E_{13}^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\frac{(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right)^{\frac{1}{4}}}_{N_9} \\ &\quad + \underbrace{\frac{C(\gamma-1)^{\frac{1}{16}} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \left(\frac{(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho}E_0}{\mu} \right)^{\frac{3}{4}}}_{N_{10}} \end{aligned}$$

$$\begin{aligned}
& \underbrace{+\frac{C(\bar{\rho}, \tilde{\rho})}{\mu} \left(\frac{(\gamma-1)^{\frac{1}{13}} E_0^{\frac{25}{36}} E_{12}}{\mu^{\frac{12}{13}}} + \frac{\tilde{\rho} E_0}{\mu} \right)}_{N_{11}} \\
& \underbrace{+\frac{C E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \left(\frac{(\gamma-1)^{\frac{1}{4}} E_0}{\mu^3} + \frac{A_1^{\frac{3}{2}}(T) A_2^{\frac{1}{2}}(T)}{\mu^3} + \frac{(\gamma-1)^{\frac{1}{6}} E_0^{\frac{2}{3}} A_1(T)}{\mu^4} + \frac{P(\tilde{\rho})^2 E_0}{\mu^3} \right)}_{N_{12}}^{\frac{1}{2}} \quad (4.70)
\end{aligned}$$

Following a process similar to N_1 - N_7 , one gets N_8 - N_{12} as follows:

$$\begin{aligned}
N_8 + N_9 = N_1 + N_2 &\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{18} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} E_{18}, \quad (4.71)
\end{aligned}$$

$$\begin{aligned}
N_{10} + N_{11} = N_3 + N_4 &\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{19} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} E_{19} \quad (4.72)
\end{aligned}$$

and

$$\begin{aligned}
N_{12} &\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} \left\{ \frac{(\gamma-1)^{\frac{1}{9}} E_0^{\frac{7}{8}}}{\mu^{\frac{11}{6}}} \right. \\
&\quad \left. + \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{32}} \frac{E_0^{\frac{1}{2}}}{\mu^2} + \frac{(\gamma-1)^{\frac{5}{72}} E_0^{\frac{17}{24}} A_1(T)}{\mu^{\frac{7}{3}}} + \frac{\tilde{\rho}^{\gamma-\frac{1}{12}} E_0^{\frac{7}{8}}}{\mu^{\frac{11}{6}}} \right\} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} \left\{ \left(\frac{(\gamma-1) E_0^{\frac{63}{8}}}{\mu^{12}} \right)^{\frac{1}{9}} \frac{1}{\mu^{\frac{1}{2}}} + \left(\frac{\tilde{\rho}}{2C} \right)^{\frac{1}{32}} \frac{E_0^{\frac{1}{2}}}{\mu^2} \right. \\
&\quad \left. + \left(\frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{32}} \frac{E_0^{\frac{1}{2}}}{\mu^2} + \left(\frac{(\gamma-1) E_0^{\frac{51}{5}}}{\mu^{12}} \right)^{\frac{5}{72}} \frac{1}{\mu^{\frac{1}{2}}} + \left(\frac{\tilde{\rho} E_0^{\frac{3}{22}}}{\mu^{\frac{22}{7}}} \right)^{\frac{7}{12}} \tilde{\rho}^{\gamma-\frac{2}{3}} \right\} \\
&\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} \left\{ \frac{1}{\mu^{\frac{1}{2}}} + \left(\frac{\tilde{\rho} E_0^{16}}{\mu^{12}} \right)^{\frac{1}{32}} \frac{1}{\mu^{\frac{13}{8}}} + \left(\frac{\tilde{\rho} E_0^{\frac{195}{2}}}{\mu^2} \right)^{\frac{1}{192}} \frac{1}{\mu^2} + \tilde{\rho}^{\gamma-\frac{2}{3}} \right\}
\end{aligned}$$

$$\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} E_{21}, \quad (4.73)$$

where

$$E_{21} = \frac{1}{\mu^{\frac{1}{2}}} + \frac{1}{\mu^{\frac{13}{8}}} + \frac{1}{\mu^2} + \tilde{\rho}^{\gamma-\frac{2}{3}}, \quad (4.74)$$

and we have used $\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \frac{\tilde{\rho}}{2C}$.

Substituting (4.71), (4.72) and (4.73) into (4.70), we obtain

$$\begin{aligned} A_1(T) &\leq C \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{1}{2}} (E_{18} + E_{19} + E_{21}) \\ &\leq \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{8}}, \end{aligned} \quad (4.75)$$

provided $\left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right) \leq \left(C(E_{18} + E_{19} + E_{21}) \right)^{-8}$. □

Now we are in a position to close the *a priori* assumption on ρ .

Lemma 4.8 *Under the conditions of Proposition 4.1, it holds that*

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \frac{7\bar{\rho}}{4} \quad (4.76)$$

for any $(x, t) \in \mathbb{R}^3 \times [0, T]$, provided

$$\frac{\left((\gamma-1)^{\frac{1}{36}} + \tilde{\rho}^{\frac{1}{6}} \right) E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \varepsilon \triangleq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}.$$

Proof. In fact, the proof is similar to the one in Lemma 3.9, then we just list some differences. Here we rewrite (3.69) as follows:

$$|b(t_2) - b(t_1)| \leq \frac{C(\bar{\rho})}{\mu^{\frac{3}{4}}} \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{16}}. \quad (4.77)$$

Therefore, for $t \in [0, \sigma(T)]$, one can choose N_0 and N_1 in Lemma 2.3 as follows:

$$N_1 = 0, \quad N_0 = \frac{C(\bar{\rho})}{\mu^{\frac{3}{4}}} \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{16}},$$

and $\bar{\zeta} = 0$. Then

$$g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = 0 \quad \text{for all } \zeta \geq \bar{\zeta} = 0.$$

Thus

$$\sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^\infty} \leq \max\{\bar{\rho}, 0\} + N_0 \leq \bar{\rho} + \frac{C(\bar{\rho})}{\mu^{\frac{3}{4}}} \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^{\frac{3}{16}} \leq \frac{3\bar{\rho}}{2}, \quad (4.78)$$

provided

$$\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4 \right\}. \quad (4.79)$$

On the other hand, for $t \in [\sigma(T), T]$, we can rewrite (3.72) as follows:

$$|b(t_2) - b(t_1)| \leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^2, \quad (4.80)$$

provided $\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \varepsilon_6$. Therefore, one can choose N_1 and N_0 in Lemma 2.3 as

$$N_1 = \frac{1}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho}) \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^2.$$

Note that

$$g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = -\frac{1}{2\mu + \lambda} \quad \text{for all } \zeta \geq 1,$$

one can set $\bar{\zeta} = 1$. Thus

$$\begin{aligned} \sup_{\sigma(T) \leq s \leq T} \|\rho\|_{L^\infty} &\leq \max \left\{ \frac{3}{2} \bar{\rho}, 1 \right\} + N_0 \leq \frac{3}{2} \bar{\rho} + C(\bar{\rho}) \left(\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \right)^2 \\ &\leq \frac{7\bar{\rho}}{4}, \end{aligned} \quad (4.81)$$

provided

$$\frac{(\gamma-1)^{\frac{1}{36}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{6}} E_0^{\frac{1}{4}}}{\mu^{\frac{1}{3}}} \leq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}. \quad (4.82)$$

The combination of (4.78) and (4.81) completes the proof of Lemma 4.8. \square

Now, the proof of Proposition 4.1 is completed. Next, following a process similar to that in the proof of Theorem 1.1, we can prove that the results obtained in Proposition 4.1 still hold in the case of $\gamma \geq 2$. At last, we will derive the time-dependent higher norm estimates of the smooth solution (ρ, u) . In fact, the proofs are the same as the ones in [16]. For the convenience, we omit them here.

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